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Discrete Probability of Random Möbius Groups: Random Subgroups by Two Generators

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Abstract: Let $GL(2,\mathbb{R})$ be real Möbius groups. For any 2×2 matrix A in $GL(2,\mathbb{R})$ induces real Möbius transformations g by the formula $A\to g_A=g$ where $A=\begin{pmatrix} a & b \\ c & d\end{pmatrix}, g=\frac{ax+b}{cx+d}$. The collection of all real Möbius transformations for which $ad-bc=1$ takes the values 1 forms a group which can be identified with $PSL(2,\mathbb{R})$. We write $f\in {}_*PSL(2,\mathbb{R})$ to mean that f is a random variable in $PSL(2,\mathbb{R})$. In this paper, we study the random Möbius subgroup ${}_*PSL(2,\mathbb{R})$. We can get some new results: (1) If $f,g\in {}_*PSL(2,\mathbb{R})$, the probability $|tr^2(f)-4|+|tr(fgf^{-1}g^{-1})-2|\le 1$ is greater than 0.027282. (2) If f is hyperbolic transformation and g is parabolic transformation, then the probability $\langle f,g\rangle$ is discrete is greater than 7/12. (3) If f is elliptic of order n and g is elliptic of order 2, then the probability $\langle f,g\rangle$ is discrete is greater than $2/n$. (4) The probability that random chosen $f,g\in {}_*PSL(2,\mathbb{R})$ generate an elementary or non-discrete group $\langle f,g\rangle$ is greater than 0.0302049.

Keywords: Random Discrete Möbius Group, ${}_*PSL(2,\mathbb{R})$, Jørgensen's Inequality

1. Introduction

Throughout this paper, we will adopt the same notations and definitions as [2, 13, 14, 15, 16] such as, discrete group, Fuchsian group, elementary group, ${}_*PSL(2,\mathbb{R})$ and so on. For example, if G is discrete and if $X, A_1, A_2 \dots$ are in G with $A_n\to X$ then $A_n=X$ for all sufficiently large n. A group G of Möbius transformation is a Fuchsian group if and only if there is some $G-$ invariant disc in which G acts discontinuously. A subgroup G of $PSL(2,\mathbb{C})$ is said to be elementary if and only if there exists a finite G-orbit in $\mathbb{R}^3$. See [2, 3, 4, 5, 6, 7, 8, 9, 10，11，12, 18, 19, 20] etc. for more details.

It is well known that Random group, introduced in the 1943 by Auerbach [1], has been extensively studied over the group of random symmetric group [4] and random finite group [11, 12]. G. J. Martin and G. O'Brier [13, 14] introduce a geometrically natural probability measure on the group $PSL(2,\mathbb{R})$. They studied the probability that random Möbius subgroup $\langle f,g\rangle$ is discrete and given a lower bound for the probability a group generated by two random hyperbolic elements or parabolic elements of ${}_*PSL(2,\mathbb{R})$ is discrete.

Theorem MO1 [13, Theorem 11.6] The probability that two randomly chosen hyperbolic transformation $f,g\in$. ${}_*PSL(2,\mathbb{R})$ generate a discrete group $\langle f,g\rangle$ is at least $2/5$.

Theorem MO2 [13, Theorem 10.3] The probability that two randomly chosen parabolic transformation $f,g\in$. ${}_*PSL(2,\mathbb{R})$ generate a discrete group $\langle f,g\rangle$ is at least $1/6$.

We further study the probability a group generated by a hyperbolic element and a parabolic element of ${}_*PSL(2,\mathbb{R})$ is discrete and obtain the following theorems.

Theorem 1.1 Let f, g be random chosen hyperbolic element and parabolic element in ${}_*PSL(2,\mathbb{R})$. Then the probability $\langle f, g\rangle$ is discrete is at least $7/12$.

G. J. Martin and G. O'Brier studied the probability a group generated by a elliptic element of order n and a hyperbolic element of ${}_*PSL(2,\mathbb{R})$ is discrete.

Theorem MO3 [13, Theorem 12.5] Let $f=\xi z$ and $\xi^n=1$. Let $g\in {}_*PSL(2,\mathbb{R})$ be a randomly chosen hyperbolic. Then the probability $\langle f, g\rangle$ is discrete is $2/n^2$.

We further study the probability a group generated by two elliptic elements of ${}_*PSL(2,\mathbb{R})$ is discrete and obtain the following theorems.

Theorem 1.2 Let f, g be random chosen elements in ${}_*PSL(2,\mathbb{R})$. If f is elliptic of order n and g is elliptic of order 2. Then the probability $\langle f, g\rangle$ is discrete is $2/n$.

G. J. Martin [15, 16] had results such as the p.d.f for trace, Commutators, Cross ratios. We will obtain the p.d.f for $m(f)$ and have the following theorem.

Theorem 1.3 Let f be randomly chosen element of $PSL(2,\mathbb{R})$. Then the random variable $m^2(f)$ has the p.d.f

$$F(s)=\begin{cases}F_1(s) & 0<s\le 8\\ F_2(s) & s\ge 8\end{cases}$$

Where,

$$F_1(s)=\frac{1}{8\pi^2}\frac{1}{\frac{s}{8}+8}\int_1^{\frac{s}{8}+1}\sqrt{\frac{y}{\left(\frac{s}{8}+1-y\right)(2-y)(y-1)}}dy$$

$$F_2(s)=\frac{1}{8\pi^2}\frac{1}{\frac{s}{8}+1}\int_1^{2}\sqrt{\frac{y}{\left(\frac{s}{8}+1-y\right)(2-y)(y-1)}}dy.$$

Finally, by using Theorem 1.3, we will obtain the following theorems.

Theorem 1.4 Let f, g be random chosen elements in ${}_*PSL(2,\mathbb{R})$. Then the probability $|tr^2(f)-4|+|tr(fgf^{-1}g^{-1})-2|\le 1$ is greater than 0.0272821.

Theorem 1.5 Let f, g be random chosen elements in ${}_*PSL(2,\mathbb{R})$. Then the probability $\langle f, g\rangle$ is non-discrete or elementary is greater than 0.0302049.

2. Preliminaries

2.1. Random Möbius Groups

If $A\in PSL(2,\mathbb{C})$ has the form

$$A=\pm\begin{pmatrix} a & c\\ \bar{c} & \bar{a}\end{pmatrix},\ |a|^2-|c|^2=1. \tag{1}$$

Then the group of all matrices satisfying (1) will be denoted $\mathcal{F}$. It is not difficult to construct an algebraic isomorphism $\mathcal{F}\equiv PSL(2,\mathbb{R})\equiv Isom^+(\mathbb{H}^2)$.

The probability distribution. The probability space is $(\mathcal{F},\mu_{\mathcal{F}})$, the space of matrices with the following imposed distribution of the entries of an element of $\mathcal{F}$.

(i) $\xi=a/|a|$ and $\eta=c/|c|$ are chosen uniformly in the circle $\mathbb{S}$, with arclength measure.

(ii) $t=|a|\ge 1$ is chosen so that

$$2\arcsin(1/t)\in[0,\pi]$$

is uniformly distributed.

Lemma 2.1.1 [13, lemma 2.3] *The random variable* $|a|\in[1,\infty)$ has the p.d.f

$$F_{|a|}(x)=\frac{2}{\pi}\frac{1}{x\sqrt{x^2-1}}.$$

we write $arg(a)\in_\mu[0,2\pi]_{\mathbb{R}}$ if $arg(a)$ uniformly distributed on $\mathbb{R}\bmod 2\pi$. The following lemma will be crucial.

Lemma 2.1.2 [13, lemma 2.4] If $arg(a), arg(b)\in {}_\mu[0,2\pi]_{\mathbb{R}}$, then $arg(ab)$, $arg(a/b)\in_\mu[0,2\pi]_{\mathbb{R}}$. Hence

$$arg(a^k)=k\,arg(a)\in {}_\mu[0,\ 2\pi]_{\mathbb{R}} \text{ for } k\in\mathbb{Z}.$$

For any

$$A=\begin{pmatrix} a & c\\ \bar{c} & \bar{a}\end{pmatrix}\in\mathcal{F} \tag{2}$$

$$|tr(A)|=|2Re(a)|=|2a\cdot cos(\theta)|,\theta=arg(a). \tag{3}$$

Using the obvious symmetries, by lemma 2.1, we may calculate

$$Pr[\{|tr(A)|>2\}]=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}(1-\frac{2}{\pi}\theta)d\theta=\frac{1}{2} \tag{4}$$

Corollary 2.1 [13, corollary 2.4] Let $f\in\mathcal{F}$ be a Möbius transformation chosen random from the distribution described in (i) and (ii). Then the probability that f is hyperbolic is equal to $1/2$, f is elliptic is equal to $1/2$ and f is parabolic is equal to 0.

See [13, Section 2] for more details.

2.2. Jørgensen's Inequality

In 1976, Jørgensen proved a necessary condition for a non-elementary two generator subgroup $PSL(2,\mathbb{C})$ to be discrete, which is called Jørgensen's inequality.

Theorem J [21, lemma 1] *If* $f, g \in PSL(2,\mathbb{C})$ generate a discerte and nonelementary group $\langle f, g\rangle$. Then

$$|tr^2(f)-4|+|tr(fgf^{-1}g^{-1})-2|\geq 1 \quad (5)$$

For each $f \in \mathcal{F}$, we let $m^2(f)$ denote the norm

$$m^2(f)=\left\|A-A^{-1}\right\|^2=2|a-\bar{a}|^2+8|c|^2 \quad (6)$$

We have the following theorem from [9, Theorem 2.7].

Theorem FG₁ If f and g *in* $PSL(2,\mathbb{R})$. Then

$$|tr^2(f)-4|\leq \frac{1}{2}m(f)^2 \quad (7)$$

$$|tr(fgf^{-1}g^{-1})-2|\leq \frac{1}{16}m(f)^2 m(g)^2. \quad (8)$$

Both of these inequalities are sharp.

According to (7) and (8), we have

$$|tr^2(f)-4|+|tr(fgf^{-1}g^{-1})-2|\leq m(f)^2\left(\frac{1}{2}+\frac{1}{16}m(g)^2\right) \quad (9)$$

2.3. An experiment on $m^2(f)m^2(g)$

If we select two random Möbius transformations, say,

$$f=\begin{pmatrix} e^{i\phi_1}\csc(\theta_1) & e^{i\alpha_1}\cot(\theta_1) \\ e^{-i\alpha_1}\cot(\theta_1) & e^{-i\phi_1}\csc(\theta_1)\end{pmatrix}, \quad (10)$$

$$g=\begin{pmatrix} e^{i\phi_2}\csc(\theta_2) & e^{i\alpha_2}\cot(\theta_2) \\ e^{-i\alpha_2}\cot(\theta_2) & e^{-i\phi_2}\csc(\theta_2)\end{pmatrix}, \quad (11)$$

where $\theta_1, \theta_2 \in_\mu [0,\pi/2]_{\mathbb{R}}$ and $\alpha_1, \alpha_2, \phi_1, \phi_2 \in_\mu [0,2\pi]_{\mathbb{R}}$.

Then

$$m^2(f)m^2(g)= 64\left(\frac{\sin^2(\phi_1)}{\sin^2(\theta_1)}+\cot^2(\theta_1)\right)\left(\frac{\sin^2(\phi_2)}{\sin^2(\theta_2)}+\cot^2(\theta_2)\right) \quad (12)$$

We made several independent runs through about 10^7 random matrix pairs of elements to generate the histogram in Figure 1. We found the probability that

$$m^2(f)m^2(g)\geq 16\left(2\cos\left(\frac{2\pi}{7}\right)-1\right) \quad (13)$$

to be about 0.0308797.

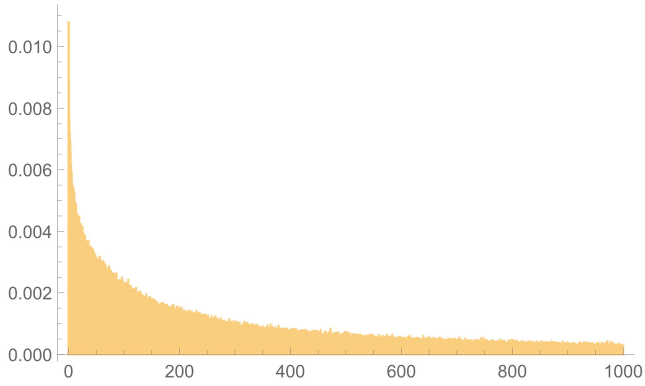

Figure 1. *Histogram of* $m^2(f)m^2(g)$ *values.*

We also found

$$\Pr\left\{m(f)^2\left(\frac{1}{2}+\frac{1}{16}m(g)^2\right)\leq 1\right\}\approx 0.027282. \quad (14)$$

According to Theorem J and Lemma 2.3, we have

$$\Pr\left\{|tr^2(f)-4|+|tr(fgf^{-1}g^{-1})-2|\leq 1\right\}\geq 0.027282 \quad (15)$$

We have the following theorem from [9, Theorem 4.19].

Theorem FG₂ Suppose that $\langle f, g\rangle$ is nonelementary discrete Fuchsian group subgroup of $\mathbb{M}$. Then

$$m^2(f)m^2(g)\geq 16(2\cos(2\pi/7)-1). \quad (16)$$

In addition, if f or g is parabolic, then

$$m^2(f)m^2(g)\geq 16. \quad (17)$$

If f and g are both elliptic, then

$$m^2(f)m^2(g)\geq 16\cos^2(\pi/7). \quad (18)$$

If f and g are both hyperbolic, then

$$m^2(f)m^2(g)\geq 16(\cos(2\pi/7)+\cos(\pi/7)-1)^2 \quad (19)$$

Finally, if f is hyperbolic and g is elliptic, then

$$m^2(f)m^2(g)\geq 16(\cos(2\pi/7)-1) \quad (20)$$

Each of these inequalities is sharp.

Another discrete criterion can be found in [9, 16].

Lemma 2.3.1 [10, Lemma 4.2] Suppose that $\langle f, g\rangle$ has parameters $(\gamma,\beta,-4)$ with $\gamma\geq\beta$ and that f is a primitive elliptic of order $q\geq 3$. Then $\langle f, g\rangle$ is a discrete group if and only if $\gamma\geq\beta+4$.

Lemma 2.3.2 [15, Theorem VII. A.13.] Let $f_i, i=1,2,\ldots,n.$ be hyperbolic transformations of the disk whose isometric disks are all disjoint. Then the group generated by these hyperbolic transformations $\langle f_1, f_2, \ldots, f_n \rangle$ is discrete.

3. Proofs of the Main Results

Proof of Theorem 1.1 Suppose that the arc $\alpha_i = \alpha\left(m_{\alpha_i}, l_{\alpha_i}\right)$ $i = 1, 2.$ of f and the arc $\beta = \beta\left(m_{\beta_i}, l_\beta\right)$ of g. Since f is hyperbolic, we have

$$\alpha_1 \cap \alpha_2 = \varnothing . \tag{21}$$

Thus

$$\alpha_1 \cap \beta = \varnothing, \alpha_2 \cap \beta = \varnothing \tag{22}$$

are independent. A little trigonometry reveals that

$$\alpha_i \cap \beta = \varnothing \Leftrightarrow \arg\left(m_{\alpha_i}, \overline{m}_\beta\right) \geq \frac{l_\alpha + l_\beta}{2} \tag{23}$$

Now the variables $\theta_i = \arg\left(m_{\alpha_i}, \overline{m}_\beta\right), i = 1, 2,$ are uniformly distributed in $[0, \pi]$ and independent.

According lemma 2.1.2, we have

$$\Pr\left\{\min \theta_i \geq \frac{l_\alpha + l_\beta}{2}\right\} = \Pr\left\{\min \theta_i - \frac{l_\beta}{2} \geq \frac{l_\alpha}{2}\right\}$$

$$= \left(\frac{1}{\pi}\int_{\frac{l_\alpha}{2}}^{\pi} d\theta\right)^2 = \left(1 - \frac{1}{\pi}\frac{l_\alpha}{2}\right)^2 .$$

Let $\frac{l_\alpha}{2} = \varphi$, then $\varphi \in_\mu \left[0, \frac{\pi}{2}\right]$, thus

$$\Pr\left\{\min \theta_i \geq \frac{l_\alpha + l_\beta}{2}\right\} = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left(1 - \frac{\varphi}{\pi}\right)^2 d\varphi = \frac{7}{12}.$$

Using lemma 2.3.2, the probability $\langle f, g \rangle$ is discrete is at least $7/12$.

Proof of Theorem 1.2 With loss of generality, f and g can be represented that

$$f = \begin{pmatrix} e^{i\frac{\pi}{n}} & 0 \\ 0 & e^{-i\frac{\pi}{n}} \end{pmatrix}, g = \begin{pmatrix} i\csc(\theta) & e^{i\varphi}\cot(\theta) \\ e^{-i\varphi}\cot(\theta) & -i\csc(\theta) \end{pmatrix} \tag{24}$$

By calculation

$$\beta = -4\sin^2\left(\frac{\pi}{n}\right), \gamma = 4\sin^2\left(\frac{\pi}{n}\right)\cot^2(\theta).$$

Using lemma 2.3.1, $\langle f, g \rangle$ is a discrete group if and only if $\gamma \geq \beta + 4$.

$$\Pr\{\gamma \geq \beta + 4\} = \Pr\left\{4\sin^2\left(\frac{\pi}{n}\right)\cot^2(\theta) \geq 4\cos^2\left(\frac{\pi}{n}\right)\right\}$$

$$= \Pr\left\{\theta \leq \frac{\pi}{n}\right\} = \frac{2}{\pi}\int_0^{\frac{\pi}{n}} d\theta = \frac{2}{n}.$$

Then the probability $\langle f, g \rangle$ is discrete is $\frac{2}{n}$.

Proof of Theorem 1.3 We select a random Möbius transformation

$$f = \begin{pmatrix} e^{i\phi}csc(\theta) & e^{i\varphi}cot(\theta) \\ e^{-i\varphi}cot(\theta) & e^{-i\phi}csc(\theta) \end{pmatrix}, \theta \in_\mu [0, \frac{\pi}{2}], \phi, \varphi \in_\mu [0, 2\pi].$$

Then

$$m^2(f) = 8\frac{\sin^2(\phi)}{\sin^2(\theta)} + 8\cot^2(\theta).$$

The probability distribution functions of $x = \sin^2(\theta)$ and $y = \sin^2(\phi)$ are independent and identically distributed $\Psi(x)$ and $\Psi(y)$,

$$\Psi(x) = \frac{1}{\pi\sqrt{x(1-x)}}$$

The random variable has the c.d.f

$$P(S \leq s) = \frac{1}{\pi^2}\iint_{\left\{8\cdot\frac{x+1-y}{y} \leq s\right\}} \frac{1}{\sqrt{x(1-x)}}\frac{1}{\sqrt{y(1-y)}}dxdy$$

We begin this calculation under the assumption $0 \leq s \leq 8$.

$$P_1(s) = \frac{1}{\pi^2}\int_{\frac{1}{\frac{s}{8}+1}}^1 \int_0^{(\frac{s}{8}+1)y-1} \frac{1}{\sqrt{x(1-x)}}\frac{1}{\sqrt{y(1-y)}}dxdy$$

$$= \frac{2}{\pi^2}\int_{\frac{1}{\frac{s}{8}+1}}^1 \frac{1}{\sqrt{y(1-y)}} \cdot arcsin[\sqrt{(\frac{s}{8}+1)y-1}]dy \tag{25}$$

$$= \frac{2}{\pi^2}\int_1^{\frac{s}{8}+1} \frac{arcsin\sqrt{y-1}}{\sqrt{y(\frac{s}{8}+1-y)}}dy$$

We can now differentiate (26) of s under the integral.

$$F_1(s) = \frac{1}{8\pi^2}\int_{\frac{1}{\frac{s}{8}+1}}^1 \sqrt{\frac{y}{(1-y)[2-(\frac{s}{8}+1)y][(\frac{s}{8}+1)y-1)]}}dy$$

$$= \frac{1}{8\pi^2}\frac{1}{\frac{s}{8}+1}\int_1^{\frac{s}{8}+1} \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}}dy \tag{26}$$

Similarity, suppose that $s \geq 8$, we can calculate

$$P_2(s) = \frac{2}{\pi^2}\int_1^2 \frac{arcsin\sqrt{y-1}}{\sqrt{y(\frac{s}{8}+1-y)}}dy + 1 - \frac{2}{\pi}\cdot arcsin[\sqrt{\frac{2}{\frac{s}{8}+1}}]$$

and

$$F_2(s) = \frac{1}{8\pi^2}\frac{1}{\frac{s}{8}+1}\int_1^2 \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}}dy.$$

The proof is completed.

A discussion similar to the proof of Theorem 1.3, we obtain some corollaries.

Corollary 3.1 Let f be randomly chosen parabolic element of ${}_*PSL(2,\mathbb{R})$, then the random variable $m^2(f)$ has the p.d.f

$$H(s) = \frac{4}{\pi}\frac{1}{\sqrt{s}(16+s)}.$$

Corollary 3.2 Let f be randomly chosen hyperbolic element of ${}_*PSL(2,\mathbb{R})$, then the random variable $m^2(f)$ has the p.d.f

$$G(s) = \frac{1}{4\pi^2}\frac{1}{\frac{s}{8}+1}\int_1^{\frac{\frac{s}{4}+2}{\frac{s}{8}+2}} \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}}dy.$$

Corollary 3.3 Let f be randomly chosen elliptic element of ${}_*PSL(2,\mathbb{R})$, then the random variable $m^2(f)$ has the p.d.f

$$T(s) = \begin{cases} T_1(s) & 0 < s \leq 8 \\ T_2(s) & s \geq 8 \end{cases}$$

Where,

$$T_1(s) = \frac{1}{4\pi^2}\frac{1}{\frac{s}{8}+1}\int_{\frac{\frac{s}{4}+2}{\frac{s}{8}+2}}^{\frac{s}{8}+1} \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}}dy$$

$$T_2(s) = \frac{1}{4\pi^2}\frac{1}{\frac{s}{8}+1}\int_{\frac{\frac{s}{4}+2}{\frac{s}{8}+2}}^{2} \sqrt{\frac{y}{(\frac{s}{8}+1-y)(2-y)(y-1)}}dy.$$

Corollary 3.4 Let f be randomly chosen element of ${}_*PSL(2,\mathbb{R})$. Then the random variable $\frac{1}{2}+\frac{1}{16}m^2(f)$ has the p.d.f

$$F(t) = \begin{cases} F_1(t) & \frac{1}{2} < t \leq 1 \\ F_2(t) & t \geq 1 \end{cases}$$

Where,

$$F_1(t) = \frac{1}{\pi^2 t}\int_1^{2t} \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}}dy,$$

$$F_2(t) = \frac{1}{\pi^2 t}\int_1^{2} \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}}dy.$$

Corollary 3.5 Let f be randomly chosen parabolic element of ${}_*PSL(2,\mathbb{R})$. Then the random variable $\frac{1}{2}+\frac{1}{16}m^2(f)$ has the p.d.f

$$H(t) = \frac{1}{\pi}\cdot\frac{1}{(t+\frac{1}{2})\sqrt{t-\frac{1}{2}}}.$$

Corollary 3.6 Let f be randomly chosen hyperbolic element of ${}_*PSL(2,\mathbb{R})$. Then the random variable $\frac{1}{2}+\frac{1}{16}m^2(f)$ has the p.d.f

$$G(t) = \frac{2}{\pi^2 t}\int_1^{\frac{4t}{2t+1}} \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}}dy.$$

Corollary 3.7 Let f be randomly chosen elliptic element of ${}_*PSL(2,\mathbb{R})$. Then the random variable $\frac{1}{2}+\frac{1}{16}m^2(f)$ has the p.d.f

$$T(t) = \begin{cases} T_1(t) & \frac{1}{2} < t \leq 1 \\ T_2(t) & t \geq 1 \end{cases}$$

Where,

$$T_1(t) = \frac{1}{2\pi^2 t}\int_{\frac{4t}{2t+1}}^{2t} \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}}dy,$$

$$T_2(t) = \frac{1}{2\pi^2 t}\int_{\frac{4t}{2t+1}}^{2} \sqrt{\frac{y}{(2t-y)(2-y)(y-1)}}dy.$$

Proof of Theorem 1.4 Suppose that f, g be randomly chosen in ${}_*PSL(2,\mathbb{R})$, we need calculate the probability of $m(f)^2(\frac{1}{2}+\frac{1}{16}m(g)^2) \leq 1$. According to Theorem 1.3 and Corollary 3.5, we have

$$Pr\left\{m(f)^2(\frac{1}{2}+\frac{1}{16}m(g)^2)\le 1\right\}=\iint_{\{s\cdot t\le 1\}}F(t)\cdot F(s)dsdt$$

$$=\int_{\frac{1}{2}}^{1}\int_{1}^{2t}\int_{1}^{\frac{1}{8t}+1}\frac{2}{\pi^4 t}\frac{arcsin\sqrt{y-1}}{\sqrt{y(\frac{1}{8t}+1-y)}}\sqrt{\frac{x}{(2t-x)(2-x)(x-1)}}dydxdt+\int_{1}^{\infty}\int_{1}^{2}\int_{1}^{\frac{1}{8t}+1}\frac{2}{\pi^4 t}\frac{arcsin\sqrt{y-1}}{\sqrt{y(\frac{1}{8t}+1-y)}}\sqrt{\frac{x}{(2t-x)(2-x)(x-1)}}dydxdt$$

$= 0.0161113 + 0.0111708 = 0.0272821.$

Using (27), we have

$$Pr\left\{m(f)^2(\frac{1}{2}+\frac{1}{16}m(g)^2)\le 1\right\}\le Pr\{|tr^2(f)-4|+|tr(fgf^{-1}g^{-1})-2|\le 1\}$$

Then

$$Pr\{|tr^2(f)-4|+|tr(fgf^{-1}g^{-1})-2|\le 1\}\ge 0.0272821.$$

Proof of Theorem 1.5 Suppose that f,g be randomly chosen in $_*PSL(2,\mathbb{R})$, we need calculate the probability of $m(f)^2m(g)^2\le 16\left(2\cos\left(2\pi/7\right)-1\right)=m$. According to Theorem 1.3, we have

$$Pr\left\{m(f)^2m(g)^2\le m\right\}$$

$$=\int_{0}^{\frac{m}{8}}\int_{1}^{\frac{t}{8}+1}\int_{1}^{2}\frac{1}{4\pi^4}\frac{1}{\frac{t}{8}+1}\frac{arcsin\sqrt{x-1}}{\sqrt{x(\frac{m}{8t}+1-x)}}\sqrt{\frac{y}{(\frac{t}{8}+1-y)(2-y)(y-1)}}dxdydt$$

$$+\int_{0}^{\frac{m}{8}}\int_{1}^{\frac{t}{8}+1}\frac{1}{8\pi^2}\frac{1}{\frac{t}{8}+1}\left(1-\frac{2}{\pi}arcsin\sqrt{\frac{2}{\frac{m}{8t}+1}}\right)\sqrt{\frac{x}{(\frac{t}{8}+1-x)(2-x)(x-1)}}dxdt$$

$$+\int_{\frac{m}{8}}^{8}\int_{1}^{\frac{t}{8}+1}\int_{1}^{\frac{m}{8t}+1}\frac{1}{4\pi^4}\frac{1}{\frac{t}{8}+1}\frac{arcsin\sqrt{x-1}}{\sqrt{x(\frac{m}{8t}+1-x)}}\sqrt{\frac{y}{(\frac{t}{8}+1-y)(2-y)(y-1)}}dxdydt$$

$$+\int_{8}^{\infty}\int_{1}^{\frac{m}{8t}+1}\int_{1}^{2}\frac{1}{4\pi^4}\frac{1}{\frac{t}{8}+1}\frac{arcsin\sqrt{y-1}}{\sqrt{y(\frac{1}{8t}+1-y)}}\sqrt{\frac{x}{(\frac{t}{8}+1-x)(2-x)(x-1)}}dxdydt$$

$$=0.00222416+0.00805606+0.0155805+0.00434425=0.0302049.$$

Using (28), we have the probability $\langle f,g\rangle$ is non-discrete or elementary is greater than 0.0302049 .

According theorem 1.5, we further have the following corollaries.

Corollary 3.8 Let f,g be random chosen elements in $_*PSL(2,\mathbb{R})$ and f or g is parabolic. Then the probability $\langle f,g\rangle$ is non-discrete or elementary is greater than 0.0881294.

Corollary 3.9 Let f,g be random chosen elements in $_*PSL(2,\mathbb{R})$ and f , g are both elliptic. Then the probability $\langle f,g\rangle$ is non-discrete or elementary is greater than 0.0756016.

Corollary 3.10 Let f,g be random chosen elements in $_*PSL(2,\mathbb{R})$ and f,g are both hyperbolic. Then the probability $\langle f,g\rangle$ is non-discrete or elementary is greater than 0.0329061.

Corollary 3.11 Let f,g be random chosen elements in $_*PSL(2,\mathbb{R})$ and f is hyperbolic and g is elliptic,. Then the probability $\langle f,g\rangle$ is non-discrete or elementary is greater than 0.0302049.

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