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Combination of Reduced Differential Transformation Method and Picard's Principle

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Abstract: In this paper, the combination of methods is used for the search for exact solutions, when they exist of mixed and non-mixed nonlinear partial differential equations. it is the combinate of reduced differential transform method and Picard principle. This combination gave us an algorithm that promotes the rapid convergence of the problem given the exact solution desired. Some complex physical behaviors can be described by mathematical expressions. These expressions can be nonlinear partial differential equations and sometimes mixed. For a better understanding of the physical phenomena associated with such partial differential equations, the exact solution, when it exists, is better indicated. However, by using classical analytical methods, the access or the obtaining of the exact solution is not always obvious. With some hybrid algorithms, the difficulties of accessing this exact solution can be difficult or almost impossible. Hence the coupling of some algorithms to reach the desired result. The objective of our work is the search for exact solutions when they exist of mixed and unmixed nonlinear partial differential equations. Although the reduced transform method has presented several interesting results, the difficulties of obtaining exact solutions have also been encountered. Thus, in this paper, a combination is used to find exact solutions, when they exist, of these types of partial differential equations. It is the combination of the reduced transform method and Picard's principle. This Picard principle, which uses the Adomian decomposition method, works as a method of successive approximations, approaching the problem by an iterative scheme. This combination gave us an algorithm that favors the fast convergence of the problem. Thus, the exact solutions of the selected problems are obtained.

Keywords: Nonlinear PDEs, Reduced Differential Transform Method (RDTM), SBA Method, Picard Principle

1. Introduction

Most physical phenomena are modeled by differential equations (ODE), integro-differential equations, partial differential equations (PDEs). Obtaining the exact solution has always been desired, when it exists. These different solutions help to understand and explain the physical phenomena that these equations model.

For the search of these exact solutions, several methods have been implemented. Depending on the nature of the equation, especially the partial differential equation, the use of classical, semi-analytical or approximation methods for the search of solutions does not always lead us to the exact solutions sought, when they exist. In order to obtain satisfactory results, algorithms of some classical methods have sometimes been improved or modified to solve some of these complex problems. With these algorithms, although modified or improved, the access to the exact solution sometimes presented considerable difficulties, to the point of being satisfied with an approximate solution.

Always with the aim of arriving at the exact solution, te researchers did not cross their arms. They have sometimes opted for the combination of methods, which has sometimes led to satisfactory results. When will refer to some cases such as: Laplace-Adomian method, Laplace-VIM method, Laplace-SBA method [15, 20], and many other. In the literature review we have gone through, we were interested in the RDTM [1-9, 11-14, 16, 17, 19, 21-25]. The efficiency of this method for solving nonlinear and mixed PDEs has impressed us. Nevertheless, we have noticed that in some cases, or for some PDEs, the access to the exact solution has proved to be difficult, to the point of being satisfied with an approximate solution. In other cases, the outcome to the exact solution was not explicit [8]. This motivated us propose a new algorithm taking into account the RDTM and the Picard principle. Our work consists in solving nonlinear and mixed PDEs by combining Picard's principle and RDTM. The Picard principle is used as in the SBA method. This combination gives us a rather simple algorithm to obtain the exact solutions, when they exist.

2. Description of the Methods

2.1. Reduced Differential Transform Method (RDTM)

In this subsection, we will give the origin of the method and then the essential elements for its application.

The Reduced Differential Transformation Method (RDTM) was first proposed by the Turkish mathematician Yildiray Keskin [12]. This method is applicable to a large class if it exists. After Yildiray Keskin and Oturanc [13, 14], The RDTM has also been used by many authors to obtain analytical approximate and in some cases exact solutions to nonlinear equations. Several types of nonlinear equations have had their different exact solutions easily obtained. We can quote the nonlinear Voltera partial integro-differential equation, the Telegraph equation The inhomogeneous nonlinear wave equation. For more details, we can refer [1-9, 17, 18, 21-24]. Nevertheless, now suppose that function of two variables $u(x,t)$ which is analytic and k-times continuously differentiable with respect to space x in the domain of our interest [17, 19]. Suppose that we can consider this function in this form: $u(x,t) = f(x).g(t)$. Based on the properties of differential transform, function can be represented as

$$u(x,t) = (\sum_{i=0}^{\infty} F(i)x^i)(\sum_{j=0}^{\infty} G(j)t^j = \sum_{k=0}^{\infty} U_k(x)t^k) \quad (1)$$

Where the function $U_k(x)$ is the transformed function of $u(x,t)$ which can be defined as:

$$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0} \quad (2)$$

From equations (1) and (2) one can deduce

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0} t^k = \sum_{k=0}^{\infty} U_k(x)t^k \quad (3)$$

In this work, the lowercase $u(x,t)$ represent the original function, while the uppercase $U_k(x)$ stand for transformed function [14, 22-24].

The details for the proper understanding of the reduced differential transformation method are well explained by Keskin who is the author [12]. Many researchers have also contributed to facilitate the understanding and use of this rich method [1-3, 6, 7, 9, 21-23].

To illustrate the basic concepts of the RDM, consider the following nonlinear partial differential equation written in an operator form:

$$Lu(x,t) + Ru(x,t) + Nu(x,t) = h(x,t) \quad (4)$$

with initial condition:

$$u(x,0) = f(x) \quad (5)$$

According to the RDTM, we can construct the iteration formula as follows [1, 3, 5, 7]:

$$(k+1)U_{k+1}(x) = G_k(x) - RU_k(x) - NU_k(x) \quad (6)$$

Some basic essential properties of the two-dimensional reduced differential transform are presented in Table below.

Table 1. *The fundamental operations of RDTM.*

Functional Form	Transformed Form
$u(x,t)$	$U_k(x) = \frac{1}{k!}\left[\frac{\partial^k}{\partial t^k} u(x,t)\right]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x)$ (α is a constant)
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k-n)$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U_{k-n}(x)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} V_r(x)U_{k-r}(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^r}{\partial t^r} u(x,t)$	$W_k(x) = (k+1)\cdots(k+r)U_{k+1}(x) = \frac{(k+r)!}{k!} U_{k+r}(x)$
$w(x,t) = \frac{\partial}{\partial x} u(x,t)$	$W_k(x) = \frac{\partial}{\partial x} U_k(x)$

2.2. Picard Principle

It is from the SBA method that we have been inspired for the technique of using the Picard principle. The principle uses the Adomian decomposition method. The method is well adapted to nonlinear PDEs [15, 18, 20].

Let us solve the following problem:

$$\begin{cases} u_t = L(u(t)) + N(u(t)), 0 < t < T \\ u(0) = f \end{cases} \quad (7)$$

in a suitable functional space V; with L a linear differential operator, N the nonlinear term, $u(t)$ the unknown function, with

$$u_t = \frac{\partial u}{\partial t}$$

By the method of successive approximations, the above problem can be approximated by the following iterative scheme:

$$\begin{cases} u_t^k = L(u^k(t)) + N(u^k(t)), 0 < t < T \\ u^k(0) = f, k = 1,2,3,\cdots \end{cases} \quad (8)$$

The resolution of the scheme (8) consists in determining at each iteration $(k = 1,2,3,\cdots)$ approximate solutions

$$u^1(t), u^2(t), u^3(t), \cdots, u^n(t), \cdots$$

But this requires a judicious choice of the initial condition u^0, and the solution U of the problem is obtained by:

$$u(t) = \lim_{k\to+\infty} u^k(t) \tag{9}$$

if $(u^k)_{k\geq 0}$ is convergent. We use the Adomian method to solve the problem at each iteration.

At the first step, $(k = 1)$ we need to know $u^0(t)$ to start the algorithm in the linear case. However, the general rule for the choice of U is to take it such that it is a solution of the equation $N(u(t)) = 0$.

Note that $u^k(0) = f, \forall\, k = 1, 2,3,\cdots$ is the Cauchy condition for each step k. The approximate solution $u^k(t)$ of (8) is

$$u(t) = \lim_{k\to\infty} u^k(t) = \lim_{k\to\infty}\left(\textstyle\sum_{n=0}^{\infty} u_n^k(t)\right) \tag{10}$$

The description below explains in an explicit way how the principle works.

u_0^1: first term of the Adomian series of step 1;

u_1^1: 2^{nd} term of the Adomian series of step 1

⋮

u_n^1: $(n+1)$ term of the Adomian series of step 1.

The approximate solution of this first iteration is written:

$$u^1(t) = \lim_{m\to+\infty} \varphi_m^1(t) \text{ where } \varphi_m^1(t) = \textstyle\sum_{n=0}^{m-1} u_n^1(t)$$

After, we will be at the second iteration from which will result the calculation of the approximate solution of this second iteration.

u_0^2: first term of the Adomian series of step 2;

u_1^2: 2^{nd} term of the Adomian series of step 2

⋮

u_n^2: $(n+1)$ term of the Adomian series of step 2.

The approximate solution of this second iteration is written:

$$u^2(t) = \lim_{m\to+\infty} \varphi_m^2(t)$$

where $\varphi_m^2(t) = \sum_{n=0}^{m-1} u_n^2(t)$

One by one, we will be at the k^{th} iteration. The approximate solution of the k^{th} iteration is written:

$$u^k(t) = \textstyle\sum_{n=0}^{m-1} u_n^k(t)$$

The solution of the problem is obtained by:

$$u(t) = \lim_{k\to+\infty} \textstyle\sum_{n=0}^{m-1} u_n^k(t)$$

Here we also show how to choose the first iteration term $u^0(t)$, of the successive approximation scheme to obtain algorithms, that converge faster to the exact solution sought by simplifying the calculations.

Just choose $u^0(t)$ such that $N(u^0(t)) = 0$.

This choice allows in fact, at the first iteration to solve only a linear problem, as shown in the process below.

$$\begin{cases} u_t^1(t) = L(u^1(t)) \\ u^1(0) = f \end{cases}$$

Where a resolution by the Adomian method allows to find $u^1(t)$. From the known $u^1(t)$, the calculation of $N(u^1(t))$ will follow.

For $k = 2$, we obtain:

$$\begin{cases} u_t^2 = L(u^2) + N(u^1) \\ u^2(0) = f \end{cases}$$

Where $u^1(t)$ is the solution obtained in the first step; and $N(u^1) = g_1$, knowing that $g_{k-1}(t) = N\left(u^{k-1}(t)\right)$. Once again, we will solve a linear problem for Adomian decomposition method. Thus, from one step to the next, the different solutions are obtained u^k, with $k = 1,2,3,\cdots$

Thus, $u(t) = \lim_{k\to+\infty} u^k(t)$.

3. Applications

3.1. Example 1

Consider the following nonlinear Fornberg-Witham equation [8, 11]. This problem has been addressed with RDTM; but the algorithm used to obtain the exact solution was not enough explicit. We have given enough details for the reader, for a better understanding.

$$u_t - u_{xxt} + u_x = uu_{xxx} - uu_x + 3u_x u_{xx} \tag{11}$$

With the initial condition: $u(x,0) = e^{\frac{1}{2}x}$

Considering the second member of equation (5) as the nonlinear part in the variable x exclusively, we obtain the following writing:

$$u_t - u_{xxt} + u_x = Nu(x,t) \tag{12}$$

If we apply the RDTM to equation (6), we obtain the following iteration formula:

$$(k+1)U_{k+1}(x) - (k+1)\left(\frac{\partial^2 U_{k+1}(x)}{\partial x^2}\right) + \frac{\partial U_k(x)}{\partial x} = NU_k(x) \tag{13}$$

Using the method of successive approximations and Picard's principle, we obtain the following algorithm (8):

$$(k+1)U_{k+1}^p - (k+1)\left(\frac{\partial^2 U_{k+1}^p}{\partial x^2}\right) + \frac{\partial U_k^p}{\partial x} = NU_k^{p-1} \tag{14}$$

From this step, $p \in \mathbb{N}^*$. Consequently, we obtain:

$$p = 1: (k+1)U_{k+1}^1 - (k+1)\left(\frac{\partial^2 U_{k+1}^1}{\partial x^2}\right) + \frac{\partial U_k^1}{\partial x} = NU_k^0 \tag{15}$$

Picard's principle consists in finding a U_0 belonging to the Hilbert space such that the nonlinear part can be cancelled. That is: $U_k^0 \in H \textit{ such that } NU_k^0 = 0$.

The following expression is derived from this.

$$(k+1)U_{k+1}^1 - (k+1)\left(\frac{\partial^2 U_{k+1}^1}{\partial x^2}\right) + \frac{\partial U_k^1}{\partial x} = 0 \quad (16)$$

we proceed to the determination of the approximate solution.

Therefore, for $k = 0$, we have:

$$U_1^1 - \left(\frac{\partial^2 U_1^1}{\partial x^2}\right) + \frac{\partial U_0^1}{\partial x} = 0 \quad (17)$$

Using the initial condition, we obtain:

$$\frac{\partial U_0^1(x)}{\partial x} = \frac{1}{2}\exp\left(\frac{1}{2}x\right) \quad (18)$$

The substitution of relation (12) in relation (11) leads us to the following inhomogeneous differential equation.

$$\frac{\partial^2 U_1^1}{\partial x^2} - U_1^1 = \frac{1}{2}\exp\left(\frac{1}{2}x\right) \quad (19)$$

Homogeneous equation:

$$\frac{\partial^2 U_1^1}{\partial x^2} - U_1^1 = 0 \quad (20)$$

Homogeneous solution: $U_{1h} = A + Be^x$

Particular solution:

$$U_{1p}^1 = \alpha\, exp\left(\frac{1}{2}x\right)$$

Either

$$\frac{1}{4}\alpha \exp\left(\frac{1}{2}x\right) - \alpha \exp\left(\frac{1}{2}x\right) = \frac{1}{2}\exp\left(\frac{1}{2}x\right) \Rightarrow \alpha = -\frac{2}{3}$$

the general solution is given by the expression:

$$U_1^1(x) = U_{1h} + U_{1p} = A + Be^x - \frac{2}{3}e^{\frac{1}{2}x} \quad (21)$$

For reasons of the initial condition, we take: $A = B = 0$.

Therefore,

$$U_1^1(x) = -\frac{2}{3}\exp\left(\frac{1}{2}x\right) \quad (22)$$

Let's proceed to the manipulation of the algorithm to obtain the different terms of the series.

$$for\ k = 1{:}\ 2U_2^1(x) - 2\frac{\partial^2 U_2^1(x)}{\partial x^2} + \frac{\partial U_1^1(x)}{\partial x} = 0 \quad (23)$$

With $\frac{\partial U_1^1(x)}{\partial x} = -\frac{1}{3}\exp\left(\frac{1}{2}x\right)$

equation (17) becomes a non homogeneous differential equation:

$$2\frac{\partial^2 U_2^1(x)}{\partial x^2} - 2U_2^1(x) = -\frac{1}{3}\exp\left(\frac{1}{2}x\right) \quad (24)$$

the characteristic equation associated with the homogeneous equation is: $r^2 + r = 0$

By the same principle of the condition imposed because of the initial condition, we obtain:

$$2A_1 \exp\left(\frac{1}{2}x\right) - \frac{1}{2}A_1 \exp\left(\frac{1}{2}x\right) = \frac{1}{3}\exp\left(\frac{1}{2}x\right)$$

Either: $A_1 = \frac{2}{9}$. So $U_{2p}^1(x) = \frac{2}{9}\exp\left(\frac{1}{2}x\right)$

$$for\ k = 2{:}\ 3U_3^1(x) - 3\frac{\partial^2 U_3^1(x)}{\partial x^2} + \frac{\partial U_2^1(x)}{\partial x} = 0 \quad (25)$$

With

$$\frac{\partial U_2^1(x)}{\partial x} = \frac{1}{9}\exp\left(\frac{1}{2}x\right) \quad (26)$$

The substitution of equation (20) in equation (19) gives the following non homogeneous equation (21) whose solution is

$$3\frac{\partial^2 U_3^1(x)}{\partial x^2} - 3U_3^1(x) = \frac{1}{9}\exp\left(\frac{1}{2}x\right) \quad (27)$$

After all calculations made, by the same principles, we obtain: $U_3^1(x) = -\frac{4}{81}\exp\left(\frac{1}{2}x\right)$

$$for\ k = 3{:}\ 4U_4^1(x) - 4\frac{\partial^2 U_4^1(x)}{\partial x^2} + \frac{\partial U_3^1(x)}{\partial x} = 0 \quad (28)$$

With

$$\frac{\partial U_3^1(x)}{\partial x} = -\frac{4}{81}\exp\left(\frac{1}{2}x\right) \quad (29)$$

By the same procedure, the differential equation is obtained and then solved. From these calculations, the solution below follows: $U_4^1(x) = \frac{2}{243}\exp\left(\frac{1}{2}x\right)$

Gold:

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)t^k = U_0(x)t^0 + U_1(x)t^1 + U_2(x)t^2 + U_3(x)t^3 + U_4(x)t^4 + \cdots$$

$$\tilde{u}_n(x,t) = \exp\left(\frac{1}{2}x\right)\left(1 - \frac{2}{3}t + \frac{2}{9}t^2 - \frac{4}{81}t^3 + \frac{2}{243}t^4 + \cdots\right)$$

$$\tilde{u}_n(x,t) = \exp\left(\frac{1}{2}x\right)\left(\sum_{k\geq 0}\frac{1}{n!}\left(-\frac{2}{3}t\right)^n\right) \quad (30)$$

The exact solution can be deduced by passing to the limit.

The exact solution is

$$u(x,t) = exp\left(\frac{1}{2}x - \frac{2}{3}t\right) \quad (31)$$

3.2. Example 2

Consider the following nonlinear equation [26] with its initial condition, given by the expression

$$\begin{cases} u_t = u^2 - 4uu_x + 2u_{xt} - \frac{1}{8}u \\ u(x,0) = exp\left(\frac{1}{4}x\right) \end{cases} \quad (32)$$

The equation can also be written as:

$$u_t - 2u_{xt} + \frac{1}{8}u = u^2 - 4uu_x \quad (33)$$

The second member of the equation is our non-linear part. This is what justifies the following writing:

$$u_t - 2u_{xt} + \frac{1}{8}u = Nu \tag{34}$$

Applying the RDTM algorithm to equation (28) above, we have the following expression:

$$(k+1)U_{k+1}(x) - 2(k+1)\frac{\partial U_{k+1}(x)}{\partial x} + \frac{1}{8}U_k(x) = NU(x) \tag{35}$$

Using the method of successive approximations and Picard's principle, we obtain the following algorithm:

$$(k+1)U_{k+1}^p(x) - 2(k+1)\frac{\partial U_{k+1}^p(x)}{\partial x} + \frac{1}{8}U_k^p(x) = NU_k^{p-1}(x) \tag{36}$$

For $p = 1$ we have:

$$(k+1)U_{k+1}^1(x) - 2(k+1)\frac{\partial U_{k+1}^1(x)}{\partial x} + \frac{1}{8}U_k^1(x) = NU_k^0(x)$$

Since, for the Picard's principle,

$$\forall \in H\ (Hilbert\ space), NU_k^0(x) = 0,$$

then:

$$(k+1)U_{k+1}^1(x) - 2(k+1)\frac{\partial U_{k+1}^1(x)}{\partial x} + \frac{1}{8}U_k^1(x) = 0 \tag{37}$$

For $k = 0$, we have:

$$U_1^1(x) - 2\frac{\partial U_1^1(x)}{\partial x} + \frac{1}{8}U_0^1(x) = 0$$

Either:

$$2\frac{\partial U_1^1(x)}{\partial x} - U_1^1(x) = \frac{1}{8}U_0^1(x) \tag{38}$$

From the homogeneous equation of (31) follows the solution:

$$U_{1h}(x) = A_h exp\left(\frac{1}{2}x\right), A_h \in \mathbb{R}$$

By analogy of previous calculations, the particular solution is:

$$U_{1p}(x) = A_p exp\left(\frac{1}{4}x\right), \quad A_p \in \mathbb{R}$$

By deriving the particular solution, substituting while taking into account the initial condition, the constant is obtained. Thus:

$$U_1^1(x) = -\frac{1}{4}exp\left(\frac{1}{4}x\right) \tag{39}$$

For $k = 1$, we have:

$$U_2^1 - 2\frac{\partial U_2^1}{\partial x} = -\frac{1}{4}U_1^1$$

Either: $2\frac{\partial U_2^1}{\partial x} - U_2^1 = -\frac{1}{32}exp\left(\frac{1}{4}x\right)$

The associated homogeneous equation is:

$$2\frac{\partial U_2^1}{\partial x} - U_2^1 = 0$$

After all calculations are done, the particular solution of this step is:

$$U_2^1(x) = \frac{1}{32}exp\left(\frac{1}{4}x\right) \tag{40}$$

$For\ k = 2$

By the same procedure, we determine $U_3^1(x)$.

$$3U_3^1(x) - 6\frac{\partial U_3^1(x)}{\partial x} + \frac{1}{8}U_2^1(x) = 0$$

Either $3U_3^1(x) - 6\frac{\partial U_3^1(x)}{\partial x} + \frac{1}{256}exp\left(\frac{1}{4}x\right) = 0$

Thus

$$3U_3^1(x) - 6\frac{\partial U_3^1(x)}{\partial x} = -\frac{1}{256}exp\left(\frac{1}{4}x\right)$$

The same hypotheses are used for the constants and the general solution of the homogeneous equation as in the previous cases. After all calculations made, which implies then the particular solution:

$$U_{3p}^1(x) = -\frac{1}{384}exp\left(\frac{1}{4}x\right) \tag{41}$$

For $k = 3$, we have:

$$8\frac{\partial U_4^1(x)}{\partial x} - 4U_4^1(x) = \frac{1}{8}U_3^1(x) \tag{42}$$

$$8\frac{\partial U_4^1(x)}{\partial x} - 4U_4^1(x) = \frac{1}{3072}exp\left(\frac{1}{4}x\right)$$

Either:

$$U_4^1(x) = \frac{1}{6144}exp\left(\frac{1}{4}x\right) \tag{43}$$

The sum of the terms gives:

$$\sum_{i\geq 0} U_i^1(t) = exp\left(\frac{1}{4}x\right) - \frac{1}{4}exp\left(\frac{1}{4}x\right) + \frac{1}{32}\ exp\left(\frac{1}{4}x\right)t^2\ - \frac{1}{384}exp\left(\frac{1}{4}x\right)t^3 + \frac{1}{6144}exp\left(\frac{1}{4}x\right)t^4 + \cdots \tag{44}$$

$$\sum_{i\geq 0} U_i^1(t) = exp\left(\frac{1}{4}x\right)\left[1 + \left(-\frac{1}{4}t\right) + \frac{1}{2!}\left(-\frac{1}{4}t\right)^2 + \frac{1}{3!}\left(-\frac{1}{4}t\right)^3 + \frac{1}{4!}\left(-\frac{1}{4}t\right)^4 + \cdots\right] \tag{45}$$

The sum of the problem is given by

$$\sum_{i\geq 0} U_i^1(t) = exp\left(\frac{1}{4}x\right) exp\left(\left(-\frac{1}{4}t\right)\right) \quad (46)$$

We check, then consequently, the exact solution is:

$$u(x,t) = exp\left(\frac{1}{4}(x-t)\right) \quad (47)$$

3.3. Example 3

Consider, for our third example, the following nonlinear equation [10] with its initial condition, given by the expression

$$\begin{cases} u_t - u_{xx} - uu_x + u_x - u + u^2 = 0 \\ u(x,0) = e^x \end{cases} \quad (48)$$

To facilitate the understanding of the rest of the work, for the identification of the nonlinear part, the fundamental equation of the problem to be solved can be written in the form:

$$u_t - u_{xx} + u_x - u = uu_x - u^2 \quad (49)$$

The equation can also be written as follows:

$$u_t - u_{xx} + u_x - u = Nu \quad (50)$$

With: $Nu = uu_x - u^2$.

As with the previous problems, if we apply the RDTM to equation (44), we obtain the following iteration formula:

$$(k+1)U_{k+1}(x) - \frac{\partial^2 U_k(x)}{\partial x^2} + \frac{\partial U_k(x)}{\partial x} - U_k(x) = NU_k(x) \quad (51)$$

We wish reiterate that, using the method of successive approximations and Picard's Principle to equation (45), we obtain the following algorithm:

$$(k+1)U_{k+1}^p(x) - \frac{\partial^2 U_k^p(x)}{\partial x^2} + \frac{\partial U_k^p(x)}{\partial x} - U_k^p(x) = NU_k^{p-1}(x) \quad (52)$$

The manipulation of the algorithm according to the of p and k gives:
For $p = 1$ we have:

$$(k+1)U_{k+1}^1(x) - \frac{\partial^2 U_k^1(x)}{\partial x^2} + \frac{\partial U_k^1(x)}{\partial x} - U_k^1(x) = NU_k^0(x) \quad (53)$$

Always: $U_k^0 \in H$, such that: $NU_k^0 = 0$
The following expression is derived from this.

$$(k+1)U_{k+1}^1(x) - \frac{\partial^2 U_k^1(x)}{\partial x^2} + \frac{\partial U_k^1(x)}{\partial x} - U_k^1(x) = 0 \quad (54)$$

For $k = 0$

$$U_1^1(x) - \frac{\partial^2 U_0^1(x)}{\partial x^2} + \frac{\partial U_0^1(x)}{\partial x} - U_0^1(x) = 0 \quad (55)$$

With $u(x,0) = e^x, \frac{\partial U_0}{\partial x} = e^x$. Therefore:

$$U_1^1(x) = e^x \quad (56)$$

For $k = 1$:

$$2U_2^1(x) - \frac{\partial^2 U_1^1(x)}{\partial x^2} + \frac{\partial U_1^1(x)}{\partial x} - U_1^1(x) = 0 \quad (57)$$

After all the calculations, we gat:

$$U_2^1(x) = \frac{1}{2}e^x \quad (58)$$

For $k = 2$:
We have the following equation

$$3U_3^1(x) - \frac{\partial^2 U_2^1(x)}{\partial x^2} + \frac{\partial U_2^1(x)}{\partial x} - U_2^1(x) = 0 \quad (59)$$

After substitution we have

$$3U_3^1(x) - \frac{1}{2}e^x + \frac{1}{2}e^x - \frac{1}{2}e^x = 0$$

Either

$$U_3^1(x) = \frac{1}{6}e^x \quad (60)$$

For $k = 3$:
We have the following equation:

$$4U_4^1(x) - \frac{\partial^2 U_3^1(x)}{\partial x^2} + \frac{\partial U_3^1(x)}{\partial x} - U_3^1(x) = 0 \quad (61)$$

After calculating the derivatives and substituting all expressions to equation (55), we have its solution:

$$U_4^1(x) = \frac{1}{24}e^x \quad (62)$$

By the same process, the following terms are obtained. Passing to the sum of all the expressions obtained gives:

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)t^k = U_0(x)t^0 + U_1(x)t^1 + U_2(x)t^2 + U_3(x)t^3 + U_4(x)t^4 + \cdots$$

Either:

$$\tilde{u}_n(x,t) = \sum_{k=0}^{n} U_k(x)t^k = e^x t^0 + e^x t^1 + \frac{1}{2}e^x t^2 + \frac{1}{6}e^x t^3 + \frac{1}{24}e^x t^4 + \cdots \quad (63)$$

By passage to the limit of the truncated series, the sequence of expressions follows:

$$\sum_{i\geq 0} U_i(x)t^k = e^x\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots\right) \quad (64)$$

$$\sum_{i\geq 0} U_i(x)t^k = e^x e^t \quad (65)$$

Therefore, the exact solution of our problem is:

$$u(x,t) = e^{x+t} \quad (66)$$

4. Conclusion

The findings of this article which focused on the exact solution of nonlinear partial differential equations, show that calculus of the combination algorithm of the Reduced Differential Transform Method method and Picard's principle is fast and it results in exact analytical solutions. Determining the exact solutions for all these systems proves the efficiency of coupled algorithm.

Obtaining exact solutions to all the problems posed, proves the effectiveness of the combination or coupling of the RDTM algorithm and Picard's principle.

References

[1] Afzal Soomro M., & Hussain J. (2019). On Study of Generalized Novikov Equation by Reduced Differential Transform Method. Indian Journal of Sciences and Technology, Vol. 12. DOI: 10.17 485/ijst/2019, ISSN (online): 0974-5645, pp: 1-6.

[2] Al-Sawoor, Ann J., Al-Amr Mohammed O. (2013). Reduced Differential Transform Method for the Generalized Ito System. International Journal of Enhanced Research in Science Technology & Engineering. ISSN: 2319-7463, Vol. 2, Issue 11, novembre-2013, pp: 135-145.

[3] Arslan D. (2021). The Approximate Solution of Singularly Perturbed Burgers-Huxley Equation with RDTM. BEU Journal of Science. 10 (3), pp: 703-709.

[4] Az-Zo'bi E., A. (2014). On the Reduced Differential Transform Method and it's application to the Generalized Burgers-Huxley Equation. Applied Mathematical Sciences. Vol. 8, N° 177, pp: 8823-8831.

[5] Behammouda B., Vazquez-Leal H. and Sarmiento-Reyes. (2014). Modified Reduced Differential Transform Method for Partial Differential-Algebraic Equations. Journal of Applied Mathematics. Vol 2014, Article ID 279481, pp: 1-9.

[6] Gashaw Belayeh, W., Obsie Mussa, Y., and Kebede Gizaw, A. (2020). Approximate Analytic Solutions of Two-Dimensional Nonlinear Klein-Gordon Equation by Using the Reduced Differential Transform Method. Mathematical Problems in Engineering. Vol. Article ID 5753974, pp: 1-12.

[7] Günerhan H. (2018). Analytical Solutions of One-Dimensional Convection-Diffusion Probems. Turkish Journal of Analysis and Number Theory. Vol. 6, N°6, pp: 152-154.

[8] Hesam, S., Nazemi, A., Haghbin A. (2012). Reduced Differential Transform Method for Solving the Fornberg-Witham Type Equation. International Journal of Nonlinear Science. Vol. 13, N°.2, pp: 158-162.

[9] Jafari H., Jassim, H., K., Moshokoa S., P., Ariyan V., Tcchier F., (2016). Reduced differential transform method for partial differential equations within local fractional derivative operators. Advances in Mechanical Engineering. Vol. 8 (4). pp: 1–6.

[10] Khalouta A. (2019). Résolution des équations aux dérivées partielles linéaires et non-linéaires moyennant des approaches analytiques. Extension aux cas d'EDP d'ordre fractionnaire. PhD diss., Université de Sétif 1-Ferhat Abbas.

[11] Karbalaie, A., Muhammed, H., H., Shabani, M., & Montazeri, M., M. (2014). Exact solution of partial differential equation using homo-separation of variables. International Journal of Nonlinear Science, ISSN 1749-3889 (print), 1749-3897, Vol. 17, N°.1, pp: 84-90.

[12] Keskin Y., Oturanc G., (2009). Reduced Differential Transform Method for Partial Differential Equations. International Journal of Nonlinear Sciences and Numerical Simulation. 10 (6), pp: 741-749.

[13] Keskin Y., (2010). Application of Reduced Differential Transformation Method for Solving Gas Dynamics Equation. Int. J. Contemp. Math. Sciences, Vol. 5, N°22, pp: 1091-1096.

[14] Keskin Y., Oturanc G. (2010). Reduced Differential Transform Method for solving linear and Nonlinear wave Equations. Iranian Journal of Science & Technology. Transaction A, Vol. 34, N°A2, pp: 113-122.

[15] Mikamona M., S., Bonazebi-Yindoula J., Paré, Y. & Bissanga G. (2019). Application of SBA method to solve the nonlinear Biological population models. European Journal of Pure and Applied Mathematics. 12 (3), 1096-1105.

[16] Mirzaee, F. (2011). Differential Transform Method for Solving Linear and Nonlinear Systems of ordinary Differential Equations. Applied Mathematical Sciences. Vol. 5, n°.70, pp: 3465-3472.

[17] Mohamed S. Mohamed, Khaled A. Gepreel. (2017). Reduced Differential Transform Method for Nonlinear integral member of Kadomtsev-Petviashvili hierarchy differential equations. Journal of the Egyptian Mathematical Society 25. pp: 1-7.

[18] Nkaya G. D., Mouyedo-Loufouilou J., Wellot Y. A. S., Bonazebi-Yindoula J. & Bissanga G. (2018). SBA Method in solving the Schrödinger equation. International J. Functional Analysis, Operator Theory and Applications. Vol. 10, n°1, pp: 1-10.

[19] Moosavi Noori S. R. & Taghizadeh N. (2020). Study on Solving Two-dimensional Linear and Nonlinear Volterra Partial Integro-differential Equations by Reduced Differential Transform Method. Applications and Applied Mathematics (AAM). Vol. 15 Issue 1, (June 2020), pp: 394-407.

[20] Paré Y., Migoungou Y. & Wassiha Nébié A. (2019). Resolution of nonlinear convection-diffusion-reaction equations of Cauchy's kind by the Laplace SBA method. European Journal of Pure and Applied Mathematics. 12 (3), 771-789.

[21] Rawashdeh M. (2013) Improved Approximate solutions for Nonlinear Evolutions Equations in Mathematical Physics Using the Reduced Differential Transform Method. Journal of Applied Mathematics & Bioinformatics, vol. 3, no. 2, pp: 1-14. ISSN: 1792-6602.

[22] Rawashdeh M., & Obeidat N. A. (2014). On finding Exact and Approximate Solutions to Some PDEs Using the Reduced Differential Transform Method. Appl. Math. Inf. Sci. Vol. 8, No. 5, pp: 2171-2176.

[23] Soltanalizadeh B. (2011). Application of Differential Transformation Method for Numerical Analysis of Kawahara Equation. Australian Journal of Basic and Applied Sciences, 5 (12). pp: 490-495. ISSN 1991-8178.

[24] Taghizadeh, N., & Moosavi oori, S. R. (2016). Exact solutions of Cubic Nonlinear Schrödinger Equation with a Trapping Potential by Reduced Differential Transform Method. Math. Sci. Lett. 5, N°3, pp: 1-5.

[25] Yildirim, K. (2012). A solution Method for Solving Systems of Nonlinear PDEs. World Applied Sciences Journal, 18 (11), 1527-1532.

[26] Zuriqat, M. (2018). The homo separation analysis method for solving the partial differential equation. Italian Journal of Pure and Applied Mathematics, n°40, pp: 535-543.