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An Integrable Nonlinear Wave Model: Darboux Transform and Exact Solutions

Jingru Geng[1], **Minna Feng**[2,*]

[1]School of Computer and Artificial Intelligence, Zhengzhou University, Zhengzhou, China
[2]School of Mathematics and Statistics, Zhengzhou University, Zhengzhou, China

Email address:
fengnma@126.com (Minna Feng)
*Corresponding author

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Abstract: Soliton equations are infinite-dimensional integrable systems described by nonlinear partial differential equations. In the mathematical theory of soliton equations, the discovery of integrability of these equations has greatly promoted the understanding of their generality, and thus promoted their rapid development. A key feature of an integrable nonlinear evolution equation is the fact that it can be expressed as the compatibility condition of two linear spectral problems, i.e., a Lax pair, which plays a crucial roles in the Darboux transformation. A major difficulty, however, is the problem of associating nonlinear evolution equations with appropriate spectral problems. Therefore, it is interesting for us to search for the new spectral problem and corresponding nonlinear evolution equations. In this paper, a new integrable nonlinear wave model and its integrable nonlinear reduction are presented by using the introduced 2×2 matrix spectral problem. Based on the resulting gauge transforms between the 2×2 matrix Lax pairs, Darboux transforms are derived for the integrable nonlinear wave model and its integrable nonlinear reduction, from which an algebraic algorithm for solving this integrable nonlinear wave model and its integrable nonlinear reduction is given. As an application of the Darboux transform, explicit exact solutions of the integrable nonlinear reduction are obtained, including solitons, breathers, and rogue waves.

Keywords: An Integrable Nonlinear Wave Model, Integrable Reduction, Darboux Transform, Exact Solutions

1. Introduction

The discovery of many solitons and integrable systems and the in-depth study of their mathematical and physical properties is one of the great advances in nonlinear science, and it has been applied in a series of scientific and technological fields. A lot of mathematical theories, such as symplectic manifold, spectral theory of differential operators, partial differential equations, Lie algebras and Lie groups and their representation theory, algebraic curves and so on, have become important tools in the study of soliton theory. In turn, the research progress of soliton theory promotes the development of these subdisciplines. Several systematic approaches have been proposed to solve these integrable nonlinear equations, for example, the inverse scattering transform [1], algebraic curve method [2, 3, 4, 12], Darboux transform [5, 6, 7, 8], and other methods [9, 10, 11]. By using these methods, explicit exact solutions to many integrable nonlinear equations are constructed, including soliton solutions, quasiperiodic solutions, breather solutions, rogue-wave solutions, peakon solutions, etc [13, 14, 15, 16, 17, 18, 19, 20].

Darboux transform (DT) is a very useful tool for solving integrable nonlinear equations. It can be used to generate new solutions from various known solutions. Furthermore, this process may proceed continuously, usually resulting in a series of exact solutions. In this article, one proposes a new integrable nonlinear wave model related to a 2×2 matrix spectral problem

$$\begin{aligned}
&u_{xt} - 2uw + iu_x + 2iuvu_x = 0, \\
&v_{xt} - 2vw - iv_x - 2iuvv_x = 0, \\
&w_t - (uv)_x = 0,
\end{aligned} \tag{1}$$

and an integrable reduction ($w = w^*, v = u^*$)

$$\begin{aligned} &u_{xt} - 2uw + iu_x + 2i(|u|^2)u_x = 0, \\ &w_t - (|u|^2)_x = 0. \end{aligned} \tag{2}$$

Then their DTs are constructed by resorting to the introduced gauge transforms between the Lax pairs and the reduction technique. As an illustrative example of applying DT, some explicit exact solutions of the integrable nonlinear reduction equation (2) are obtained, such as solitons, breathers and rogue waves.

The present paper is organized as follow. In Section 2, one first derives a Lax pair of the nonlinear wave equation (1) and then deduces a gauge transform between the Lax pairs, from which the DT of the nonlinear wave model (1) is constructed. Using the reduction technique, the DT of the integrable nonlinear reduction (2) is gained. In Section 3, appropriate parameters are chosen to get the corresponding "seed solutions". Various explicit exact solutions of the integrable nonlinear reduction (2), such as solitons, breathers and rogue waves, are given by using the DT.

2. Darboux Transforms

In the present section, one will find a Lax pair of the nonlinear wave equation (1) and construct its DT. Then, a DT of the integrable nonlinear reduction (2) is obtained through the reduction technique. For this purpose, one introduces a Lax pair, a 2×2 matrix spectral problem and an auxiliary problem,

$$\phi_x = U\phi, \quad U = \begin{pmatrix} i\lambda w & (1+\lambda)u_x \\ \lambda v_x & -i\lambda w \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{3}$$

$$\phi_t = V\phi, \quad V = -\frac{i}{2\lambda}\begin{pmatrix} 1+\lambda+2uv\lambda & -2(1+\lambda)u \\ 2v\lambda & -(1+\lambda+2uv\lambda) \end{pmatrix}, \tag{4}$$

where u, v, w are three potentials, and $\lambda \in \mathbb{C}$ is a spectral parameter independent with x and t. By direct calculation, the following conclusion holds.

Theorem 2.1. Assume that ϕ satisfies (3) and (4). Then the compatibility condition $\phi_{xt} = \phi_{tx}$ generates the zero-curvature equation, $U_t - V_x + [U, V] = 0$, which is exactly the nonlinear wave equation (1).

To derive the DT of (1), a gauge transform of the Lax pair, (3) and (4), is defined by

$$\hat{\Phi} = \mathcal{T}\Phi, \quad \mathcal{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5}$$

where

$$A = 1 + \sum_{k=1}^{N} A_k\lambda^k, \quad B = (1+\lambda)\sum_{k=1}^{N} B_k\lambda^{k-1},$$
$$C = \sum_{k=1}^{N} C_k\lambda^k, \quad D = 1 + \sum_{k=1}^{N} D_k\lambda^k.$$

Suppose that (5) transforms (3) and (4) into a Lax pair of $\hat{\Phi}$:

$$\hat{\Phi}_x = \hat{U}\hat{\Phi}, \quad \hat{\Phi}_t = \hat{V}\hat{\Phi}. \tag{6}$$

Then one infers

$$\hat{U} = (\mathcal{T}_x + \mathcal{T}U)\mathcal{T}^{-1}, \quad \hat{V} = (\mathcal{T}_t + \mathcal{T}V)\mathcal{T}^{-1}. \tag{7}$$

Substituting (5) into $\hat{U}\mathcal{T} = \mathcal{T}_x + \mathcal{T}U$, $\hat{V}\mathcal{T} = \mathcal{T}_t + \mathcal{T}V$, and comparing the coefficients of λ^j $(1 \leq j \leq N)$ yield

$$\begin{aligned} &\hat{u} = u + B_1, \quad \hat{v} = v + C_1, \\ &\hat{w} = w + i\frac{\partial}{\partial x}\ln\left(1 + \sum_{k=1}^{N}(-1)^k A_k\right) = w - i\frac{\partial}{\partial x}\left(1 + \sum_{k=1}^{N}(-1)^k D_k\right). \end{aligned} \tag{8}$$

For fixed N constants $\lambda_1, \lambda_2, \ldots, \lambda_N \in \mathbb{C}$, $\lambda_j \neq \lambda_k$ for $j \neq k$, and all $j = 1, 2, \cdots, 2N$, let $\varphi(\lambda_j) = (\varphi_1(\lambda_j), \varphi_2(\lambda_j))^T$ and $\psi(\lambda_j) = (\psi_1(\lambda_j), \psi_2(\lambda_j))^T$ be a fundamental system of solutions of (3) and (4) at $\lambda = \lambda_j$. Assuming $\alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{C}$ are some arbitrary constants, a general solution of the spectral problems (3) and (4) at λ_j is given by $\varphi(\lambda_j) + \alpha_j\psi(\lambda_j)$. Therefore, one considers the following linear algebraic system:

$$\begin{aligned}&\sum_{k=1}^{N}A_k\lambda_j^k+\sum_{k=1}^{N}B_k\beta_j\lambda_j^{k-1}(1+\lambda_j)=-1,\\&\sum_{k=1}^{N}C_k\lambda_j^k+\sum_{k=1}^{N}D_k\beta_j\lambda_j^{k}=-\beta_j,\end{aligned}\tag{9}$$

with

$$\beta_j=\frac{\varphi_2(\lambda_j)+\alpha_j\psi_2(\lambda_j)}{\varphi_1(\lambda_j)+\alpha_j\psi_1(\lambda_j)},\quad 1\le j\le N.\tag{10}$$

The constants λ_j and α_j are chosen such that the coefficient matrix of (9) is nondegenerate. Thus, A_k, B_k, C_k, D_k, $(1\le k\le N)$, are uniquely given by (9). It's easy to verify from (9) that

$$\Big(1+\sum_{k=1}^{N}A_k\lambda_j^k\Big)\Big(1+\sum_{k=1}^{N}D_k\lambda_j^k\Big)=(1+\lambda_j)\lambda_j\Big(\sum_{k=1}^{N}B_k\lambda_j^k\Big)\Big(\sum_{k=1}^{N}C_k\lambda_j^k\Big),\tag{11}$$

which means that λ_j $(1\le j\le 2N)$ are the roots of $\det\mathcal{T}$. Because $\det\mathcal{T}=1$ when $\lambda=0$, $\det\mathcal{T}$ can be written as

$$\det T=A_ND_N\prod_{j=1}^{2N}(\lambda-\lambda_j)=\prod_{j=1}^{2N}\frac{\lambda-\lambda_j}{\lambda_j}.\tag{12}$$

The above results show that $\lambda=\lambda_j$ $(1\le j\le 2N)$ are removable isolated singularities of $\hat{U}$ and $\hat{V}$. Therefore, $\hat{U}$ and $\hat{V}$ for all $\lambda\in\mathbb{C}$ can be defined by analytic continuations. If a gauge transform changes a Lax pair into another Lax pair of the same type, one calls it a DT of the integrable nonlinear equation related to the Lax pair.

Theorem 2.2. The Lax matrix $\hat{U}$ given by (7)-(10) possesses the same form as the Lax matrix U

$$\hat{U}=\begin{pmatrix}i\lambda\hat{w} & (1+\lambda)\hat{u}_x\\ \lambda\hat{v}_x & -i\lambda\hat{w}\end{pmatrix},\tag{13}$$

where new potentials $\hat{u},\hat{v},\hat{w}$ are determined by the DT (8).

Proof Take $\mathrm{adj}(\mathcal{T})=(\det\mathcal{T})\mathcal{T}^{-1}$ and Let

$$(\mathcal{T}_x+\mathcal{T}U)\,\mathrm{adj}(\mathcal{T})=\begin{pmatrix}h_{11}(\lambda) & h_{12}(\lambda)\\ h_{21}(\lambda) & h_{22}(\lambda)\end{pmatrix}.\tag{14}$$

It's obvious that $h_{sl}(\lambda)$ $(s,l=1,2)$ are polynomials of order $(2N+1)$ for λ. Noting (4), (9) and (10), one arrives at

$$\begin{aligned}&\beta_{j,x}=-(1+\lambda_j)u_x\beta_j^2-2i\lambda_jw\beta_j+\lambda_jv_x,\\&A_x(\lambda_j)=-\beta_{j,x}B(\lambda_j)-\beta_jB_x(\lambda_j),\\&C_x(\lambda_j)=-\beta_{j,x}D(\lambda_j)-\beta_jD_x(\lambda_j),\quad(1\le j\le 2N).\end{aligned}\tag{15}$$

It's easy to verify that λ_j $(1\le j\le 2N)$ are roots of $h_{sl}(\lambda)$ $(s,l=1,2)$ with the aid of (14) and (15). Thus, $(\mathcal{T}_x+\mathcal{T}U)\,\mathrm{adj}(\mathcal{T})=(\det\mathcal{T})Q(\lambda)$, where

$$Q(\lambda)=\begin{pmatrix}Q_{11}^{(1)}\lambda+Q_{11}^{(0)} & Q_{12}^{(1)}\lambda+Q_{12}^{(0)}\\ Q_{21}^{(1)}\lambda+Q_{21}^{(0)} & Q_{22}^{(1)}\lambda+Q_{22}^{(0)}\end{pmatrix},\tag{16}$$

and $Q_{kl}^{(s)}$ $(s=0,1;k,l=1,2)$ are independent of λ. Thus, one obtains

$$\mathcal{T}_x+\mathcal{T}U=Q(\lambda)\mathcal{T}.\tag{17}$$

Comparing the corresponding coefficients of λ^j $(1\le j\le N+1)$ in (17) yields

$$\begin{aligned}&Q_{11}^{(1)}=i\hat{w},\quad Q_{11}^{(0)}=0,\quad Q_{12}^{(1)}=\hat{u}_x,\quad Q_{12}^{(0)}=\hat{u}_x,\\&Q_{21}^{(1)}=\hat{v}_x,\quad Q_{21}^{(0)}=0,\quad Q_{22}^{(1)}=-i\hat{w},\quad Q_{22}^{(0)}=0.\end{aligned}\tag{18}$$

This shows that $\hat{U}$ and U have the same form.

Theorem 2.3. The Lax matrix $\hat{V}$ determined by (7)-(10) possesses the same form as the Lax matrix V

$$\hat{V} = -\frac{i}{2\lambda}\begin{pmatrix} 1+\lambda+2\hat{u}\hat{v}\lambda & -2(1+\lambda)\hat{u} \\ 2\hat{v}\lambda & -(1+\lambda+2\hat{u}\hat{v}\lambda) \end{pmatrix}, \tag{19}$$

where new potentials $\hat{u}, \hat{v}, \hat{w}$ are given by the DT (8).

Proof Take $\mathrm{adj}(\mathcal{T}) = (\det \mathcal{T})\mathcal{T}^{-1}$ and

$$(\mathcal{T}_t + \mathcal{T}V)\,\mathrm{adj}(\mathcal{T}) = -\frac{i}{2\lambda}\begin{pmatrix} l_{11}(\lambda) & l_{12}(\lambda) \\ l_{21}(\lambda) & l_{22}(\lambda) \end{pmatrix}. \tag{20}$$

It's not difficult to verify that $l_{kl}(\lambda)$, $(k, l = 1, 2)$, are polynomials of order $(2N+1)$ for λ.

From (4), (9) and (10), one gets

$$\begin{aligned} &\beta_{j,t} = -\frac{i}{2\lambda}[2(1+\lambda)u\beta_j^2 - 2(1+\lambda+2uv\lambda)\beta_j + 2v\lambda], \\ &A_t(\lambda_j) = -\beta_{j,t}B(\lambda_j) - \beta_j B_t(\lambda_j), \\ &C_t(\lambda_j) = -\beta_{j,t}D(\lambda_j) - \beta_j D_t(\lambda_j), \quad (1 \le j \le 2N). \end{aligned} \tag{21}$$

In a similar way, one can prove that $\lambda_j, (1 \le j \le 2N)$, are roots of $l_{kl}(\lambda)$, $(k, l = 1, 2)$ in terms of (20) and (21).

Therefore, $(\mathcal{T}_t + \mathcal{T}V)\,\mathrm{adj}(\mathcal{T}) = (\det \mathcal{T})R(\lambda)$, where

$$R(\lambda) = -\frac{i}{2\lambda}\begin{pmatrix} r_{11}^{(1)}\lambda + q_{11}^{(0)} & r_{12}^{(1)}\lambda + q_{12}^{(0)} \\ r_{21}^{(1)}\lambda + q_{21}^{(0)} & r_{22}^{(1)}\lambda + q_{22}^{(0)} \end{pmatrix}, \tag{22}$$

and $r_{kl}^{(s)}$, $(\, s = 0, 1; k, l = 1, 2)$, are independent of λ. Thus, (20) may be read as

$$\mathcal{T}_t + \mathcal{T}V = R(\lambda)\mathcal{T}. \tag{23}$$

By equating the coefficients of λ^j, $(-1 \le j \le N)$ in (23), one can arrive at

$$\begin{aligned} &r_{11}^{(1)} = 1 + 2\hat{u}\hat{v}, \quad r_{11}^{(0)} = 1, \quad r_{12}^{(1)} = -2\hat{u}, \quad r_{12}^{(0)} = -2\hat{u}, \\ &r_{21}^{(1)} = 2\hat{v}, \quad r_{21}^{(0)} = 0, \quad r_{22}^{(1)} = -(1+2\hat{u}\hat{v}), \quad r_{22}^{(0)} = -1. \end{aligned}$$

This completes the proof.

On the basis of Theorems 2 and 3, the transforms (5) and (8) convert the Lax pair (3) and (4) to the Lax pair (6) of the same type. Therefore, one immediately gets the following result.

Theorem 2.4. Let (u, v, w) be a solution of the nonlinear wave model (1). Then, $(\hat{u}, \hat{v}, \hat{w})$ determined by the DT (8) is a new solution of the nonlinear wave model (1), where A_k, B_k, C_k, D_k of (8) are uniquely given by the linear algebraic system (9).

In the following, the DT of the integrable reduction model (2) is discussed. Under the constraints $v = u^*, w = w^*$, (1) turn into (2), and the corresponding Lax pair becomes

$$\phi_x = U\phi, \quad U = \begin{pmatrix} i\lambda w & (1+\lambda)u_x \\ \lambda u_x^* & -i\lambda w \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{24}$$

$$\phi_t = V\phi, \quad V = -\frac{i}{2\lambda}\begin{pmatrix} 1+\lambda+2|u|^2\lambda & -2(1+\lambda)u \\ 2u^*\lambda & -(1+\lambda+2|u|^2\lambda) \end{pmatrix}. \tag{25}$$

Two solutions of(24) and (25) are chosen as

$$\varphi(\lambda) = (\varphi_1(\lambda), \varphi_2(\lambda))^T, \quad \psi(\lambda) = ((1+\lambda)\varphi_2^*(\lambda^*), \lambda\varphi_1^*(\lambda^*))^T. \tag{26}$$

Assume that

$$\lambda_{2j} = \lambda_{2j-1}^*, \quad \alpha_{2j} = \lambda_{2j}^{-1}(1+\lambda_{2j})^{-1}(\alpha_{2j-1}^*)^{-1}, \quad (1 \le j \le N). \tag{27}$$

A straightforward calculation shows that $\beta_{2j} = \lambda_{2j}(1+\lambda_{2j})^{-1}(\beta^*_{2j-1})^{-1},\ D_k = A^*_k,\ C_k = B^*_k,\ (1 \le j \le N)$, and

$$\hat{v} = v + C_1 = u^* + B^*_1 = (u + B_1)^* = \hat{u}^*, \tag{28}$$

$$\hat{w} = w + i\frac{\partial}{\partial x}\ln\biggl(1+\sum_{k=1}^{N}(-1)^k A_k\biggr) = w^* + i\frac{\partial}{\partial x}\ln\biggl(1+\sum_{k=1}^{N}(-1)^k D^*_k\biggr) \tag{29}$$

$$= \biggl[w + i\frac{\partial}{\partial x}\ln\biggl(1-\sum_{k=1}^{N}(-1)^k D_k\biggr)\biggr]^* = \hat{w}^*, \tag{30}$$

$$\beta_{2j-1} = \frac{\varphi_2(\lambda_{2j-1}) + \alpha_{2j-1}\lambda_{2j-1}\varphi^*_1(\lambda^*_{2j-1})}{\varphi_1(\lambda_{2j-1}) + \alpha_{2j-1}(1+\lambda_{2j-1})\varphi^*_2(\lambda^*_{2j-1})}, \quad (1 \le j \le N), \tag{31}$$

where A_k and B_k are determined by

$$\begin{aligned}
&\sum_{k=1}^{N} A_k\lambda^k_{2j-1} + \sum_{k=1}^{N} B_k\beta_{2j-1}\lambda^{k-1}_{2j-1}(1+\lambda_{2j-1}) = -1,\\
&\sum_{k=1}^{N} A_k\beta^*_{2j-1}(\lambda^*_{2j-1})^k + \sum_{k=1}^{N} B_k(\lambda^*_{2j-1})^k = -\beta^*_{2j-1}, \quad (1 \le j \le N).
\end{aligned} \tag{32}$$

For example, when $N = 1$ and when $N = 2$, one obtains from (32), respectively, that

$$A_1 = -\frac{(\lambda_1+1)|\beta_1|^2 - \lambda^*_1}{\lambda^*_1[(\lambda_1+1)|\beta_1|^2 - \lambda_1]}, \quad B_1 = \frac{(\lambda_1 - \lambda^*_1)\beta^*_1}{\lambda^*_1[(\lambda_1+1)|\beta_1|^2 - \lambda_1]}, \tag{33}$$

and

$$\begin{pmatrix} A_1 \\ A_2 \\ B_1 \\ B_2 \end{pmatrix} = -\begin{pmatrix} \lambda_1 & \lambda^2_1 & (1+\lambda_1)\beta_1 & \lambda_1(1+\lambda_1)\beta_1 \\ \lambda^*_1\beta^*_1 & (\lambda^*_1)^2\beta^*_1 & \lambda^*_1 & (\lambda^*_1)^2 \\ \lambda_3 & \lambda^2_3 & (1+\lambda_3)\beta_3 & \lambda_3(1+\lambda_3)\beta_3 \\ \lambda^*_3\beta^*_3 & (\lambda^*_3)^2\beta^*_3 & \lambda^*_3 & (\lambda^*_3)^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ \beta^*_1 \\ 1 \\ \beta^*_3 \end{pmatrix}. \tag{34}$$

Then, one arrives at the following result.
Theorem 2.5. Let (u, v) be a solution of the integrable reduction model (2). Assume the DT reads

$$\hat{u} = u + B_1, \quad \hat{w} = w + i\frac{\partial}{\partial x}\ln\biggl(1+\sum_{k=1}^{N}(-1)^k A_k\biggr), \tag{35}$$

where B_1 and A_k, $(1 \le k \le N)$, are determined by the system of linear algebraic equations (32). Then $(\hat{u}, \hat{w})$ given by the DT (35) is a new solution of the integrable reduction model (2).

3. Exact Solutions of the Integrable Reduction Model

In this section, exact solutions of the integrable reduction model (2) will be constructed by applying the DT (35). Substituting the seed solution $(u, v) = (0, 1)$ of (2) into (24) and (25) yields a fundamental system of solutions,

$$\varphi(\lambda) = \begin{pmatrix} e^{i\lambda x - \frac{i}{2}(1+\lambda^{-1})t} \\ 0 \end{pmatrix}, \quad \psi(\lambda) = \begin{pmatrix} 0 \\ e^{-i\lambda x + \frac{i}{2}(1+\lambda^{-1})t} \end{pmatrix}. \tag{36}$$

According to (31), one deduces

$$\beta_{2j-1} = \alpha_{2j-1}e^{-2i\lambda_{2j-1}x + i(1+\lambda^{-1}_{2j-1})t}, \quad 1 \le j \le N. \tag{37}$$

When $N = 1$ and choosing $\alpha_1 = 1 - i$ and $\lambda_1 = \frac{1}{2}(1+i)$, one obtains from (33) and (35) that

$$\hat{u} = \frac{-2ie^{(1+i)x+(1-2i)t}}{1-(2-i)e^{2x+2t}}, \quad \hat{w} = \frac{1+5e^{4x+4t}}{1-4e^{2(x+t)}+5e^{4(x+t)}}. \tag{38}$$

See Figure 1 for an illustration of (38).

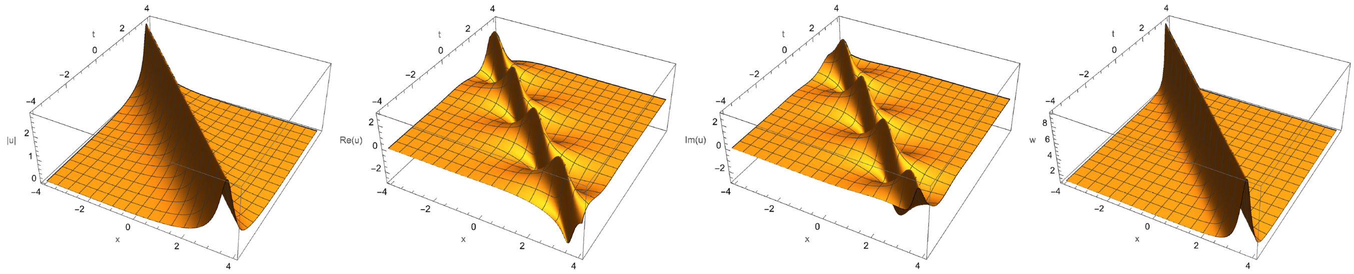

Figure 1. *A one-soliton solution.*

For $N = 2$, assume $\alpha_1 = 1$, $\alpha_3 = 1$, $\lambda_1 = 1 + i$. $\lambda_3 = \frac{1}{2}(1 + i)$. A two-soliton solution is obtained by using (32), (31) and (35), which is illustrated in Figure 2.

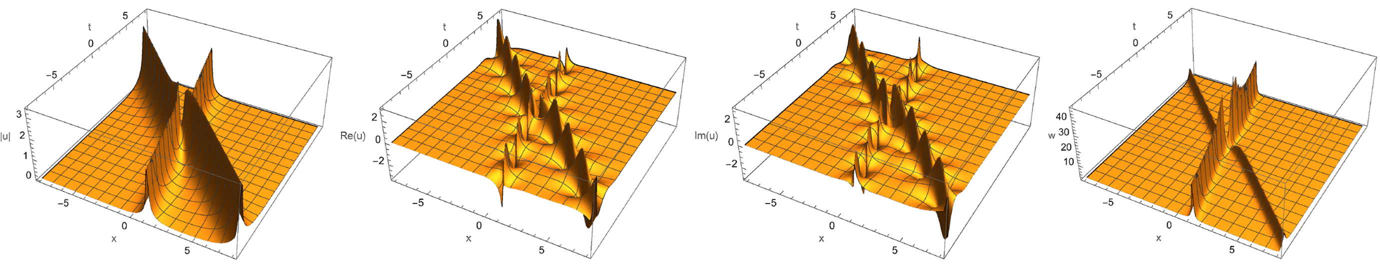

Figure 2. *A two-soliton solution.*

Choosing $\alpha_1 = 1$, $\alpha_3 = 1$, $\lambda_1 = 1+i$ and $\lambda_3 = -1+i$, and using the DT (35), one arrives at a breather solution. See Figure 3 for an illustration.

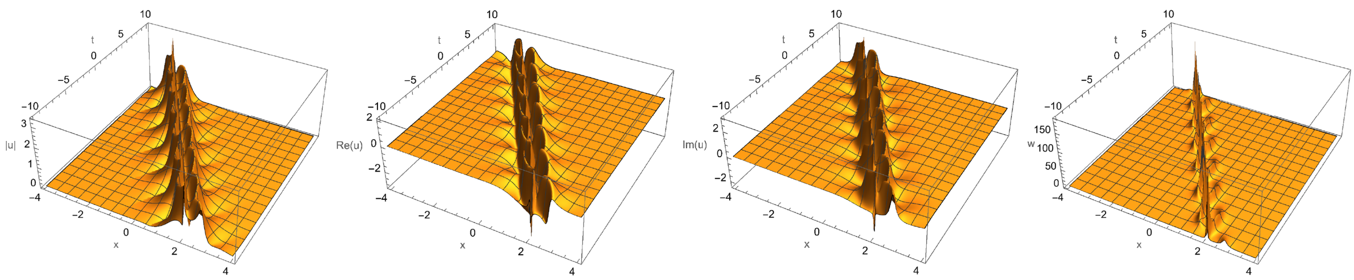

Figure 3. *A breather solution.*

Let the seed solution be $u = e^{ix}$ and $w = -\frac{3}{2}$, and choose the parameter $\lambda_1 = -\frac{1}{5} + \frac{2i}{5}$. Then, from the spectral problems (24) and (25), one deduces

$$\varphi(\lambda_1) = \begin{pmatrix} -i[(1+2i) + (1-3i)t + (1+i)x]e^{ix/2} \\ [-(1+2i)t + x]e^{-ix/2} \end{pmatrix}, \quad \psi(\lambda_1) = \begin{pmatrix} (1-i)e^{ix/2} \\ e^{-ix/2} \end{pmatrix}. \tag{39}$$

Choosing $\alpha_1 = 0$ implies from (32), (31) and (35) that

$$\begin{aligned} \hat{u} &= e^{ix}\left(-1 + \frac{10 + (8-8i)x}{5 + (10+10i)t^2 + 6x + (2+2i)x^2 - 10t - (4+4i)xt}\right), \\ \hat{w} &= -\frac{3}{2} + 2\frac{\partial}{\partial x}\arctan\left(\frac{2x^2 - 4xt + 10t^2}{2x^2 - 4xt + 6x + 10t^2 - 10t + 5}\right). \end{aligned} \tag{40}$$

This is a rogue-wave solution as illustrated in Figure 4.

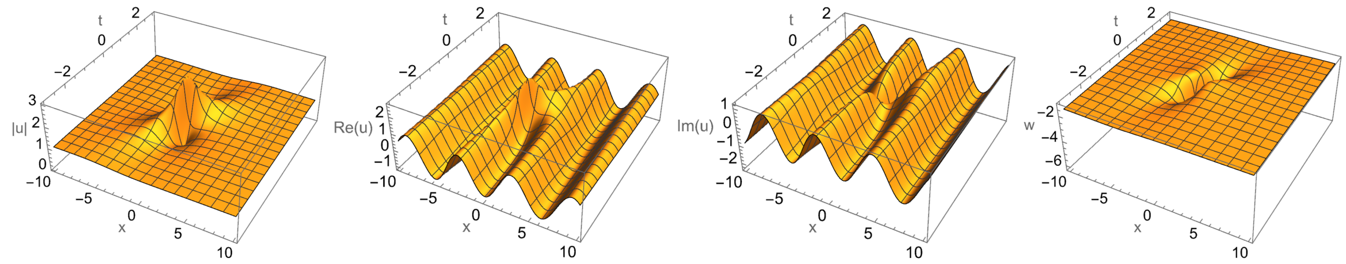

Figure 4. A rogue-wave solution.

4. Conclusions

In the present paper, one introduces a 2×2 matrix spectral problem and derives a new integrable nonlinear wave system. The system is in itself interesting and is simplified to a novel integrable complex nonlinear wave equation. One finds the gauge transform between the Lax pairs of the integrable nonlinear wave system and constructs its Darboux transforms. Through the reduction technique, Darboux transforms of the integrable nonlinear reduction equation are obtained by analysing the symmetries of the Lax pair. On this basis, an algebraic algorithm is given to solve the integrable nonlinear wave system and its integrable nonlinear reduction. As an illustrative example of our method, some explicit exact solutions of the integrable nonlinear reduction equation are constructed by resorting to the resulting Darboux transform and Mathematica software, including soliton solutions, breather solutions, rogue-wave solutions. These results are very convenient for application and analysis. In addition to this, one knows even less about the integrable nonlinear wave system and its integrable nonlinear reduction. Whether the integrable nonlinear wave system and its integrable nonlinear reduction have Bäcklund transform, conserved quantity, Hamiltonian structure and other properties is still a problem to be solved and will be discussed later.

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