Applied and Computational Mathematics
2023; 12(5): 114-139
http://www.sciencepublishinggroup.com/j/acm
doi: 10.11648/j.acm.20231205.12
ISSN: 2328-5605 (Print); ISSN: 2328-5613 (Online)

Trapping Issues for Weight-dependent Walks in the Weighted Extended Cayley Networks

Dandan Ye[1, 2, †], Fei Zhang[1, 2, †], Yiteng Qin[3], Xiaojuan Zhang[1, 2], Ning Zhang[1, 2], Jin Qin[4], Wei Chen[1, 2], Yingze Zhang[1, 2, 5, *]

[1]Institute of Orthopaedics, the Third Hospital of Hebei Medical University, Shijiazhuang, China
[2]Key Laboratory of Biomechanics of Hebei Province, Shijiazhuang, China
[3]Hebei Medical University Basical Medicine College, Shijiazhuang, China
[4]Hebei Orthopedic Clinical Research Center, Shijiazhuang, China
[5]Chinese Academy of Engineering, Beijing, China

Email address:

yedandan0408@163.com (Dandan Ye), zhangyingze_zy@126.com (Yingze Zhang)
[*]Corresponding author
[†]Dandan Ye and Fei Zhang are co-first authors.

To cite this article:

Dandan Ye, Fei Zhang, Yiteng Qin, Xiaojuan Zhang, Ning Zhang, Jin Qin, Wei Chen, Yingze Zhang . Trapping Issues for Weight-dependent Walks in the Weighted Extended Cayley Networks. *Applied and Computational Mathematics*. Vol. 12, No. 5, 2023, pp. 114-139.
doi: 10.11648/j.acm.20231205.12

Received: August 31, 2023; **Accepted:** September 25, 2023; **Published:** November 1, 2023

Abstract: The weighted extended Cayley networks are an extension of extended Cayley networks, which are the structures constructed by introducing power spaces into traditional Cayley trees. The weighted extended Cayley networks are constructed depending on two structural parameters of the network m, n and a weight factor r. Firstly, we used a new calculation method to calculate the exact analytic formula of the average weighted shortest path (AWSP). The obtained results show that: (1) For very large systems, the AWSPs for different value of weight factor r are less affected by the parameter m. (2) The AWSPs are less affected by the weight factor r when r is greater than 0 and less than or equal to n, while the AWSPs depend on the scaling factor r when r is greater than n. We have presented a trapping issue of weight-dependent walks in the weighted extended Cayley networks, focusing on a specific case with a perfect trap located at the central node. Then, the scaling expression of the average trapping time (ATT) is derived based on the layering of weighted extended Cayley networks. It was surprisingly found that (1) Regardless of the relationship between m and n, the dominant terms of ATTs are consistent. (2) ATTs are less affected by the structural parameter m and the weight factor r when r is less than or equal to the ratio of n to $m-1$, indicating that the efficiency of the trapping process is independent of m and r. (3) When r is greater than the ratio of n to $m-1$, the efficiency of the trapping process depends on three main parameters: two structural parameters of the networks m, n and a weight factor r, which means that the smaller the multiplier of three numbers r, n and $m-1$ is, the more efficient the trapping process is. Therefore, the trapping efficiency of the weighted extended Cayley networks is not only affected by the underlying structures of the networks m and n, but also by the weight factor r.

Keywords: Trapping Time, Average Weighted Shortest Path, Weighted Extended Cayley Networks, Weight-dependent Walk

1. Introduction

In recent years, complex network have attracted attentions from various fields, including physics, chemistry, biology and computer science [1]. A large number of scientists in various fields are increasingly interested in transforming molecular, fractal and other structures into complex network. These structures include various lattices [2–7], Sierpinski gasket [8–10], Sierpinski tower [11], Koch curve [12–15], T-fractals and their extensions[16–21], pseudofractal scale-free web [22–25], Apollonian packing [26–29], modular graphs [30–32], as well as macromolecules (dendrimers[33–35], hyperbranched polymers[36–40], etc). Dendrimers and regular hyperbranched polymers are two classic families of macromolecules, which can be modeled by Cayley trees and Vicsek fractals [41–44], respectively. The main reason is that complex networks provide a good way to analyze their topological characteristics and dynamic processes.

Weighted networks have attracted increasing attention in various scientific fields[45, 46]. Weighted networks are extension of networks or graphs, in which each edge between nodes i and j is associated with a variable w_{ij}, called the weight[47, 48]. Taking the ecological networks for example, the intensity of predator-prey interactions can reflect the importance of edges in ecosystems[49]. Therefore, weighted networks not only reflect practical significance in real life, but also have high theoretical significance. The heterogeneity of weights affects dynamical processes taking place on the weighted networks. Carletti et al first defined a special class of weighted fractal networks[46]. Dai et al constructed weighted scale-free networks, weighted modular networks, and studied the impact of weights on the topological characteristics and dynamical processes of their networks[47, 48, 50].

Among various dynamical processes, random walks are related to other dynamics, which include transport in media [51], disease spreading [52], target search [53], and so on. There are three most general kinds of random walks: standard random walk, weight-dependent walk and strength-dependent walk [54]. For the classic random walk, a particle may choose one of its neighboring edges at the same probability in the unweighted networks. For the weight-dependent walk or strength-dependent walk, the particle will choose an edge according to its weight or the strength of the node connected by it in the weighted networks. Compared with standard random walks, weight-dependent walk not only affects the topological structure of weighted networks, but also has a significant impact on the dynamical processes in weighted networks. The weight-dependent walk is useful for studying the topological structure of weighted networks, such as the average weighted shortest path. In particular, as an integral theme of random walks, trapping is related to a wide variety of contexts [12], such as photon-harvesting processes in photosynthetic cells [55], lighting harvesting in antenna systems [33, 56, 57], energy or exciton transport in polymer systems [58, 59], and so on.

In the weighted networks, trapping process refers to the absorption of all particles accessing a deep trap positioned at a given location with the weight-dependent walk. A basic quantity related to the trapping problem is the trapping time (TT), commonly known as the mean first-passage time (MFPT) for weight-dependent walk[44]. The MFPT is defined as the expected time for a particle starting off from a source point to arrive at the trap for the first time. Then, The average trapping time (ATT), which is defined as the average of trapping time over all starting nodes, is usually used as an important indicator to measure the trapping efficiency. Dai et al studied the trapping time for weight-dependent walk in the weighted pseudofractal scale-free networks[60], the weighted scale-free triangulation networks[61], and the weighted tetrahedron Koch networks[62], respectively. In this paper, we construct the weighted extended Cayley networks depending on two structural parameters of the network m, n and a weight factor r and study the influence of the parameters m, n, r on the topological characteristics and dynamical processes of the weighted extended Cayley networks.

The paper is organized as follows. In Section 2, we construct a model of the weighted extended Cayley networks depending on two structural parameters of the network m, n and a weight factor r. In Section 3, the average weighted shortest path (AWSP) is computed based on this division of weighted extended Cayley networks. Then, in Section 4, we calculate the the average trapping time (ATT) with weighted-dependent walk in the weighted extended Cayley networks. Finally, a conclusion is given in Section 5.

2. The Weighted Extended Cayley Networks

The purpose of this section is to construct the weighted extended Cayley networks. Intuited by Cayley trees[19, 43, 44], extended networks[37, 63] and weighted networks[46, 64], we can construct the weighted extended Cayley networks in an iterative manner.

Let us fix a positive real numbers $r > 1$, a positive integer $m \geq 3$ and a positive integer $0 \leq n \leq m-1$. Denote by $C_g(m,n)$ the weighted extended Cayley networks after g iterations. Then, the weighted extended Cayley networks are built as follows.

1. At iteration $g = 0$, $C_0(m,n)$ consists of an isolated node, called the central node.
2. At iteration $g = 1$, m new nodes are generated connecting the central node to form $C_1(m,n)$. Let $C_1(m,n)$ be our base graph, composed by $m+1$ nodes and m edges with unit weight. The $m+1$ nodes in $C_1(m,n)$ are labeled by $0, 1, 2, \cdots, m$. $C_1(m,n)$ has m segments (chains), which are expressed as $C_1^{1,0}(m,n), C_1^{2,0}(m,n), \cdots, C_1^{m,0}(m,n)$, respectively. Each segment has two nodes $0, j (j = 1, 2, \cdots, m)$ and a edge with unit weight.
3. At iteration $g = 2$, $C_2(m,n)$ is obtained from $C_1(m,n)$: Let $C_1^{(k)}(m,n)(k = 1,2,3)$ be made up of $m-1$ segments of $C_1(m,n)$, whose weighted edges have been scaled by the weight factor r. The central node $0'$ of $C_1^{(k)}(m,n)(k = 1,2,3)$ is recorded as the labeled node $j(j = 1, 2, \cdots, m)$. The labeled node $j(j = 1, 2, \cdots, m)$ of $C_1^{(k)}(m,n)(k = 1,2,3)$ is linked to the central node 0 with $n^{2-2} = n^0 = 1$ edge of unit weight.
Remark: $C_1^{(k)}(m,n)(k = 1,2,3) = C_1^{1,0}(m,n) \cup C_1^{2,0}(m,n) \cup \cdots \cup C_1^{m-1,0}(m,n) = C_1^{2,0}(m,n) \cup C_1^{3,0}(m,n) \cup \cdots \cup C_1^{m,0}(m,n) = C_1^{3,0}(m,n) \cup C_1^{4,0}(m,n) \cup \cdots \cup C_1^{m,0}(m,n) \cup C_1^{1,0}(m,n)$.
4. At iteration $i(1 < i < g)$, $C_i(m,n)$ is obtained from $C_{i-1}(m,n)$: Let $C_{i-1}^{(k)}(m,n)(k = 1,2,3)$ be made up of $m-1$ segments of $C_{i-1}(m,n)$, whose weighted edges have been scaled by the weight factor r. The

central node $0'$ of $C_{i-1}^{(k)}(m,n)(k=1,2,3)$ is recorded as the labeled node $j(j=1,2,\cdots,m)$. The labeled node $j(j=1,2,\cdots,m)$ of $C_{i-1}^{(k)}(m,n)(k=1,2,3)$ is linked to the central node 0 with n^{i-2} edges of unit weight.

Remark: $C_{i-1}^{(k)}(m,n)(k=1,2,3) = C_{i-1}^{1,0}(m,n) \cup C_{i-1}^{2,0}(m,n) \cup \cdots \cup C_{i-1}^{m-1,0}(m,n) = C_{i-1}^{2,0}(m,n) \cup C_{i-1}^{3,0}(m,n) \cup \cdots \cup C_{i-1}^{m,0}(m,n) = C_{i-1}^{3,0}(m,n) \cup C_{i-1}^{4,0}(m,n) \cup \cdots \cup C_{i-1}^{m,0}(m,n) \cup C_{i-1}^{1,0}(m,n)$. Figure 1 illustrates the division of the weighted extended Cayley networks $C_g(4,2)$ at the iteration $g=3$ when $m=4$ and $n=2$. Figure 2 illustrates the construction process of the weighted extended Cayley networks $C_g(4,2)$ from $g=1$ to $g=3$.

5. At the last iteration g, $C_g(m,n)$is obtained from $C_{g-1}(m,n)$ (see Figure 3): Let $C_{g-1}^{(k)}(m,n)(k=1,2,3)$ be made up of $m-1$ segments of $C_{g-1}(m,n)$, whose weighted edges have been scaled by the weight factor r. The central node $0'$ of $C_{g-1}^{(k)}(m,n)(k=1,2,3)$ is recorded as the labeled node $j(j=1,2,\cdots,m)$. The labeled node $j(j=1,2,\cdots,m)$ of $C_{g-1}^{(k)}(m,n)(k=1,2,3)$ is linked to the central node 0 with n^{g-2} edges of unit weight. Thus, the weighted extended Cayley networks is set up.

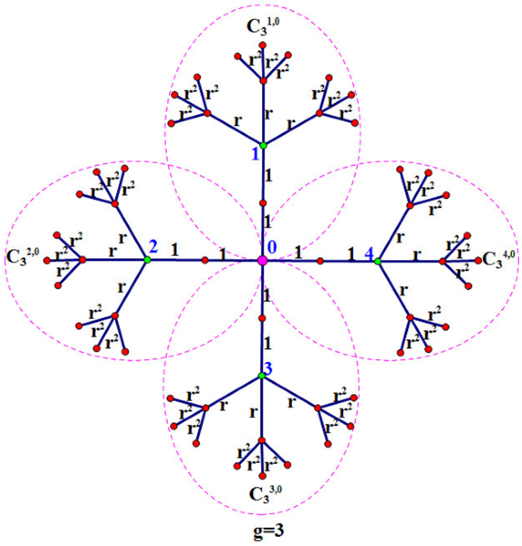


Figure 1. *$C_g(4,2)$ is regarded as merging $C_g^{1,0}(4,2), C_g^{2,0}(4,2), C_g^{3,0}(4,2), C_g^{4,0}(4,2)$ for $g=3$. The pink dot represents the center node, while the green dots represent the labeled nodes.*

This paper stipulates that: $0^{i-2}=i-1$, so this special network is consistent with the extended dendrimers[63] when $m=3$, $n=0$ and $r=1$. To calculate some intermediate parameters, we introduce this special extended dendrimers. When $n=1$ and $r=1$, the network is the Cayley trees[43].

Based on the construction of the weighted extended Cayley network $C_g(m,n)$, all nodes at the iteration g can be divided into M_g+1 levels, where M_g is defined as the minimum number of edges from an arbitrary peripheral node to the center. Then, it is easy to verify that

$$M_g = \sum_{j=0}^{g-2} n^j = \begin{cases} g & , \text{ if } \quad n=1, \\ \dfrac{n^{g-1}+n-2}{n-1}, & \text{if} \quad n>1. \end{cases} \tag{1}$$

More specifically, the central node 0 is located at level 0 and the peripheral nodes are located at level M_g.

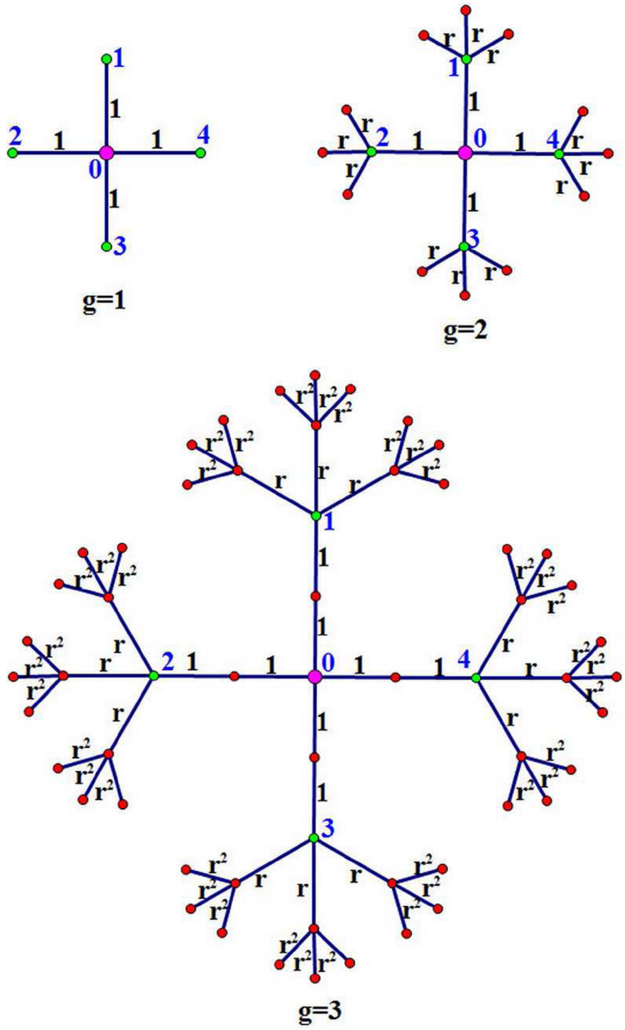


Figure 2. *Iterative construction method for the weighted extended Cayley networks $C_g(4,2)$ from $g=1$ to $g=3$. The pink dot represents the center node, while the green dots represent the labeled nodes.*

$G_i(g)$ is defined as the iteration at which all nodes at level i are created. Then

$$G_i(g) = \begin{cases} 0 & , \text{ if } \quad i=0, \\ j|j \in (M_g - M_{g+1-j}, M_g - M_{g-j}], & \text{if} \quad 0<i<M_g, \\ g & , \text{ if } \quad i=M_g. \end{cases} \tag{2}$$

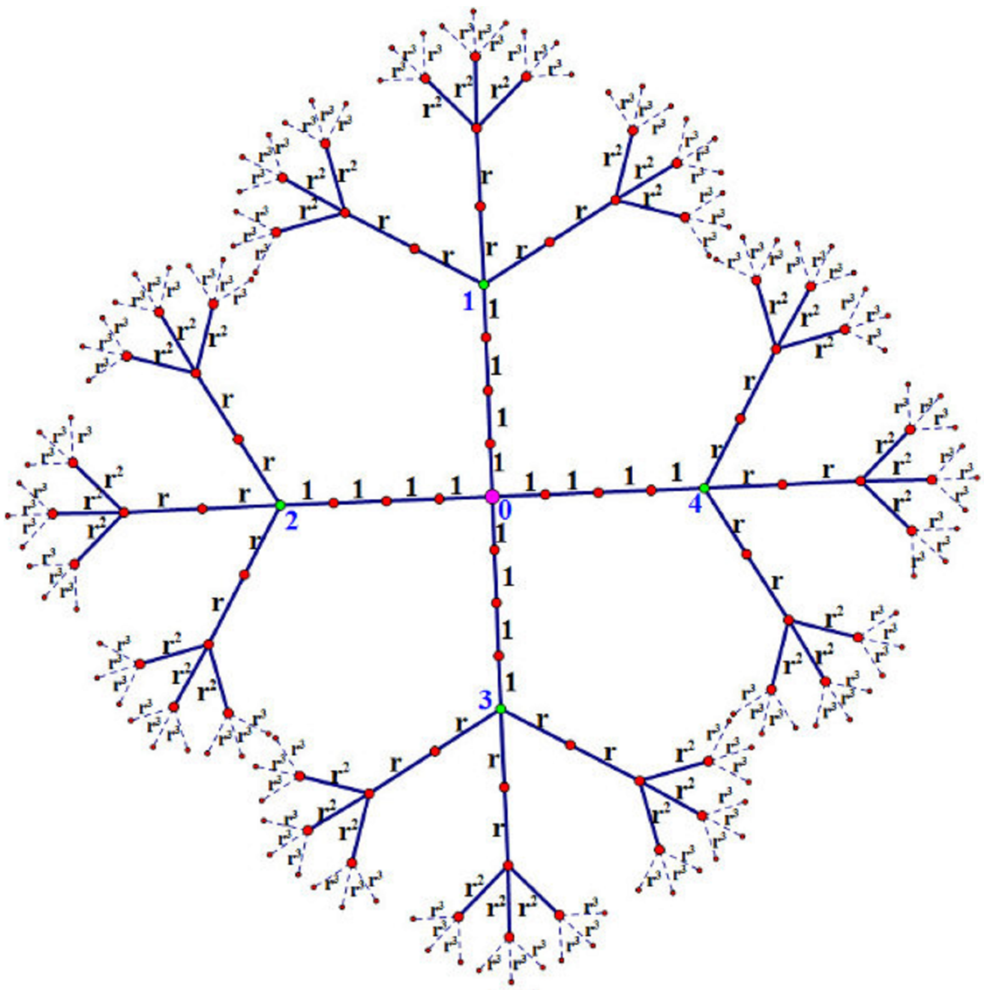


Figure 3. Structure of an weighted extended Cayley network $C_4(4,2)$ when $m = 4$ and $n = 2$. The pink dot represents the center node, while the green dots represent the labeled nodes.

$L_i(g)$ represents the value of levels i at which the degree of nodes changes, and $L_i(g)$ is defined to be

$$L_i(g) = \begin{cases} M_g - M_{g-i}, & \text{if} \quad 0 \le i < g, \\ M_g \quad , & \text{if} \quad i = g. \end{cases} \tag{3}$$

Let $N_i(g)$ denote the number of nodes at level i. According to the construction approach, it is easy to derive that $N_i(g)$ is

$$N_i(g) = \begin{cases} 1 \quad , & \text{if} \quad i = 0, \\ m \times (m-1)^{G_i(g)-1}, & \text{if} \quad 1 \le i \le M_g. \end{cases} \tag{4}$$

Let N_g denote the total number of nodes at the generation g. By construction, the derivation formula for N_g is

$$N_g = m \sum_{i=0}^{g-2} (m-1)^i \cdot n^{g-2-i} + m(m-1)^{g-1} + 1. \tag{5}$$

Then Eq. (5) can be solved to yield

$$N_g = \begin{cases} (n+1)(g+n-1)n^{g-2} + 1 \quad , & \text{if} \quad n = m-1, \\ m \times \dfrac{(m-n)(m-1)^{g-1} - n^{g-1}}{m-1-n} + 1, & \text{if} \quad n \neq m-1, \end{cases} \tag{6}$$

and the total number of edges in C_g is $E_g = N_g - 1$.

3. Average Weighted Shortest Path of the Weighted Extended Cayley Networks

The aim of this section is to determine average weighted shortest path of the weighted extended Cayley networks. Average weighted shortest path (AWSP) is defined as the average of the weighted shortest path between two nodes over all node pairs[46, 65–67]. The AWSP of the weighted extended Cayley networks $C_g(m,n)$ is given by

$$\overline{d}_g = \frac{D_{tot}(g)}{N_g(N_g-1)/2}, \tag{7}$$

where

$$D_{tot}(g) = \sum_{i,j\in C_g(m,n)} d_{ij}(g). \tag{8}$$

Let $d_{ij}(g)$ represent the weighted shortest path linking nodes i and j in $C_g(m,n)$.

In order to calculate the accurate numerical value of $D_{tot}(g+1)$, we will split the weighted extended Cayley network $C_{g+1}(m,n)$ into $m+1$ groups, which are $C_g^{(0)}(m,n), C_g^{(1)}(m,n), \cdots, C_g^{(m)}(m,n)$. Each group $C_g^{(j)}(m,n)(j=1,2,\cdots,m)$ is $m-1$ segments of $C_g(m,n)$, whose weighted edges have been scaled by a weight factor r. The labeled node $j(j=1,2,\cdots,m)$ of $C_g^{(j)}(m,n)(j=1,2,\cdots,m)$ is linked to the central node 0 with n^{g-1} edges of unit weight. The group $C_g^{(0)}(m,n)$ have $m\cdot n^{g-1}+1$ nodes, including m labeled nodes of $C_g^{(j)}(m,n)(j=1,2,\cdots,m)$. Figure 4 shows the division of the weighted extended Cayley network $C_3(4,2)$ when $m=4$ and $n=2$.

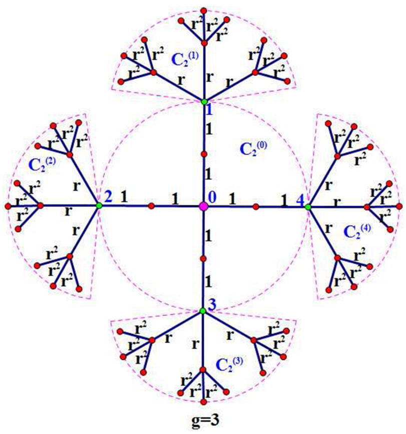


Figure 4. *The division method of the weighted network $C_3(4,2)$ is to divide it into 5 branches, namely $C_2^{(0)}(4,2)$, $C_2^{(1)}(4,2)$, $C_2^{(2)}(4,2)$, $C_2^{(3)}(4,2)$ and $C_2^{(4)}(4,2)$.*

Through this division of $C_{g+1}(m,n)$, $D_{tot}(g+1)$ satisfies the iterative formula as follows:

$$\begin{aligned} D_{tot}(g+1) &= \sum_{i,j\in C_g^{(1)}(m,n)} d_{ij} + \sum_{i,j\in C_g^{(2)}(m,n)} d_{ij} + \cdots + \sum_{i,j\in C_g^{(m)}(m,n)} d_{ij} + \sum_{i,j\in C_g^{(0)}(m,n)} d_{ij} + \Omega_g, \\ &= m\sum_{i,j\in C_g^{(1)}(m,n)} d_{ij} + \sum_{i,j\in C_g^{(0)}(m,n)} d_{ij} + \Omega_g. \end{aligned} \tag{9}$$

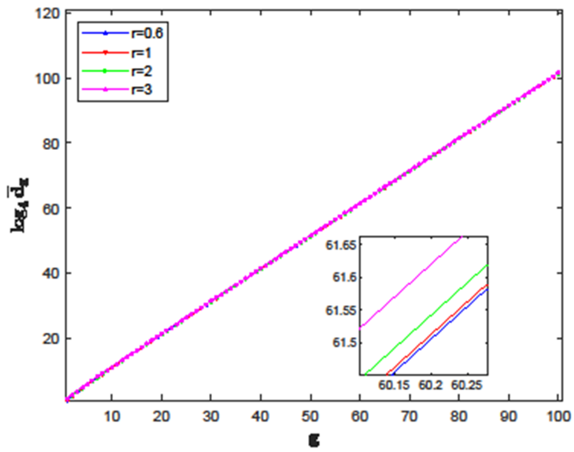
where Ω_g is the sum over all weighted shortest paths whose nodes are not in the same copy of $C_g^{(j)}(m,n)(j=0,1,2,\cdots,m)$. Note that the weighted paths that contribute to Ω_g must all go through the central node 0. Thus, the problem of determining $D_{tot}(g+1)$ is reduced to calculating $\sum_{i,j\in C_g^{(0)}(m,n)} d_{ij}$, Ω_g and $\sum_{i,j\in C_g^{(1)}(m,n)} d_{ij}$, (see Appendix).

For $n=m-1$, Eq. (9) can be obtained as

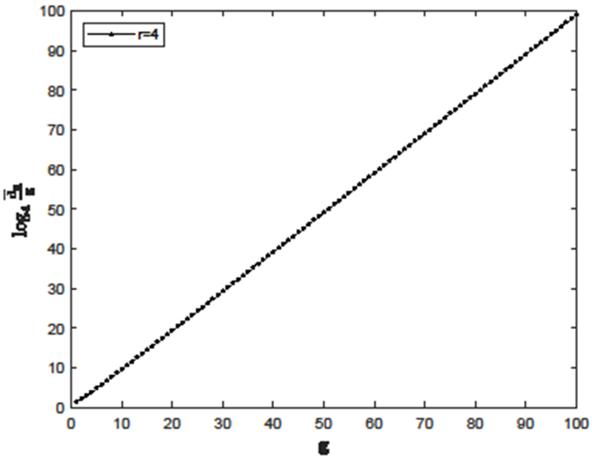
$$\begin{aligned} D_{tot}(g+1) &= (n+1)\sum_{i,j\in B_g} d_{ij} + [(ng+n^2+1)n^{g-1}+1]\cdot nr\cdot[(n+1)(nr)^{g-1} + \frac{n+1}{n^4} \\ &\times \sum_{i=0}^{g-2}(g-i)(n^2)^{g-i}(nr)^i + \frac{(n+1)(2n-3)}{2n^4}\times\sum_{i=0}^{g-2}(n^2)^{g-i}(nr)^i + \frac{n+1}{2n^2}\sum_{i=0}^{g-2}n^{g-i}(nr)^i] \\ &+ (n+1)(g+n-1)^2n^{3g-2} + (n+1)(g+n-1)\times\frac{(3n+1)n^{3(g-1)}+(n+1)n^{2(g-1)}}{2} \\ &+ \frac{(n+1)(3n+1)n^{3(g-1)}}{6} + \frac{(n+1)^2n^{2(g-1)}}{2} + \frac{(n+1)n^{g-1}}{3}. \end{aligned} \tag{10}$$

Inserting Eq. (10) into Eq. (7), $\overline{d}_g$ can be obtained as

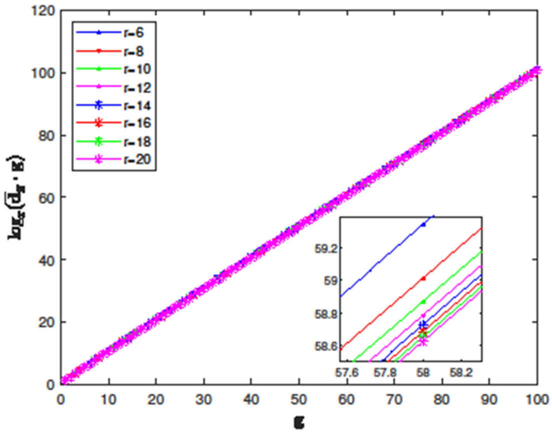
$$\overline{d}_g \sim \begin{cases} n^g, & \text{if} \quad r<n, \\ g\cdot n^g, & \text{if} \quad r=n, \\ \dfrac{r^g}{g}, & \text{if} \quad r>n, \end{cases} \tag{11}$$



(a) $\log_4 \overline{d}_g$ vs. g.



(b) $\log_4 \frac{\overline{d}_g}{g}$ vs. g.



(c) $\log_r(\overline{d}_g \cdot g)$ vs. g.

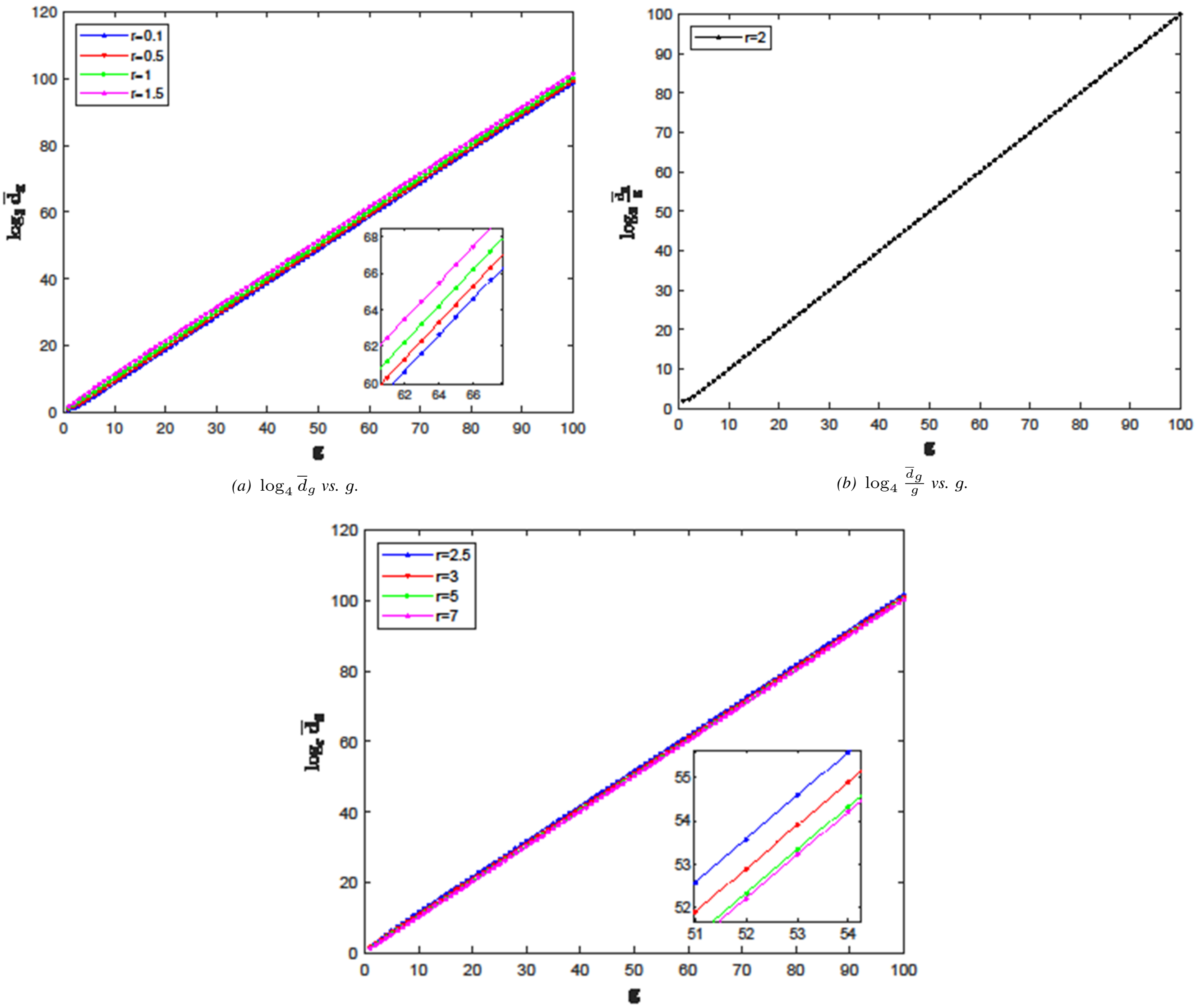
Figure 5. *The average weighted shortest path for different value of weight factor r when $m = 5$ and $n = 4$.*

For $n \neq m-1$, Eq. (9) can be obtained as

$$\begin{aligned} D_{tot}(g+1) &= m\sum_{i,j\in B_g} d_{ij} + [m\cdot n^{g-1} + 1 + \frac{(m-n)(m-1)^{g+1} - (m-1)^2\cdot n^{g-1}}{m-1-n}] \times (mr-r)[m(mr-r)^{g-1} \\ &+ \frac{m(m-1)(m-n)}{m-1-n} \times \sum_{i=0}^{g-2}(mn-n)^{g-2-i}(mr-r)^i - \frac{m(m+n-1)}{2(m-1-n)} \times \sum_{i=0}^{g-2}(n^2)^{g-2-i}(mr-r)^i \\ &+ \frac{m}{2}\sum_{i=0}^{g-2} n^{g-2-i} \times (mr-r)^i] + \frac{(m-1)^3\cdot n^{g-1}}{(m-1-n)^2}[(m-n)(m-1)^{g-1} - n^{g-1}]^2 \\ &+ m(m-1)\cdot\frac{1}{2}[(3m-2)n^{2(g-1)} + m\cdot n^{g-1}]\cdot\frac{(m-n)(m-1)^{g-1} - n^{g-1}}{m-1-n} \\ &+ \frac{(3m^2-2m)n^{3g-3}}{6} + \frac{m^2\cdot n^{2g-2}}{2} + \frac{m\cdot n^{g-1}}{3}. \end{aligned} \tag{12}$$

Inserting Eq. (12) into Eq. (7), $\overline{d}_g$ can be obtained as

$$\overline{d}_g \sim \begin{cases} n^g, & \text{if} \quad r < n, \\ g\cdot n^g, & \text{if} \quad r = n, \\ r^g, & \text{if} \quad r > n, \end{cases} \tag{13}$$



(a) $\log_4 \overline{d}_g$ vs. g.

(b) $\log_4 \frac{\overline{d}_g}{g}$ vs. g.

(c) $\log_r(\overline{d}_g)$ vs. g.

Figure 6. *The average weighted shortest path for different value of weight factor r when $m = 5$ and $n = 2$.*

For very large systems, the average weighted shortest paths (AWSPs) for different value of weight factor r are less affected by the parameter m. When $0 < r \leq n$, the dominant terms of the AWSPs are consistent according to Eqs. (11) and (13), regardless of the relationship between m and n . when $0 < r < n$, the AWSP grows with the g power of n in Figures 5(a) and 6(a). When $r = n$, $\frac{\overline{d}_g}{g}$ scales with the g power of n in Figures 5(b) and 6(b). Therefore, the AWSPs are less affected by the weight factor r when $0 < r \leq n$. When $n = m - 1$, $\overline{d}_g \cdot g$ for $r > n$ grows as a power-law function of the weight factor r with the exponent, represented by the network iteration g (see Figure 5(c)). When $n \neq m - 1$, the AWSP for $r > n$ grows as a power-law function of the weight factor r with the exponent g (see Figure 6(c)). Thus, the weighted extended Cayley networks cannot show the characteristics of the small-world systems.

4. Average Trapping Time of the Weighted Extended Cayley Networks

The main focus of this section is to study the trapping problem of weight-dependent walks with a single trap located on the central node in the weighted extended Cayley networks C_g. The purpose of our research on trapping problems is to explore the impact of network structural characteristics and weights on trapping efficiency.

The trapping problem of weight-dependent walks in the weighted networks requires introducing some basic concepts, including the weight, the strength of the node and the transition probability from the node to its neighbor node. The weight is defined as a variable w_{ij} assigned to the edges between nodes i and j in the weighted network [68, 69]. The variable w_{ij} between nodes i and j is assigned by the nth power of r, which is called the weight factor. The strength of the node is the sum of edge weights of all edges connected to the node[70]. For weight-dependent walk, a walker chooses one of its nearest neighbors with probability proportional to the weight of edge linking them[54, 71]. The transition probability from node i to its neighbor j is

$$P^{w}_{i\to j}=\frac{w_{ij}}{s_i}=\frac{w_{ij}}{\sum_{j\in\nu(i)}w_{ij}}, \tag{14}$$

where s_i is the strength of node i and $\nu(i)$ is the set of neighbor nodes of node i.

Next, we will introduce a parameter indicator (the average trapping time) to evaluate and describe many important dynamic processes of the weighted network, such as trapping efficiency. The trapping time (TT) is known as the mean first-passage time (MFPT), which is the expected time for a walker starting off from a source point to first reach the trap[72, 73]. Let T_{ij} be the TT from the node i to the trap node j. The average trapping time (ATT) is defined as the average of trapping time over all starting nodes. $\langle T\rangle_g$ is represented as the ATT to the trap positioned on the central node in the weighted extended Cayley networks C_g. By definition, $\langle T\rangle_g$ is given by

$$\langle T\rangle_g=\frac{1}{N_g-1}\sum_{i=1}^{N_g}T_{ij}(g)=\frac{T_{tot}(g)}{N_g-1}, \tag{15}$$

where $T_{tot}(g)$ represents the sum of the TTs for all nodes to the trap located on the central node.

Let $F_i(g)$ denote the TT for a node at level i in C_g. Based on the layering of C_g, $F_i(g)$ is given by

$$F_i(g)=\begin{cases} 0, \text{ if } \quad i=0,\\ \dfrac{1}{(m-1)r+1}[F_{i-1}(g)+1]+\dfrac{(m-1)r}{(m-1)r+1}[F_{i+1}(g)+1], \text{ if } \quad 0<i<M_g \;\; and \;\; i=L_{G_i(g)}(g),\\ \dfrac{1}{2}[F_{i-1}(g)+1]+\dfrac{1}{2}[F_{i+1}(g)+1], \text{ if } \quad 0<i<M_g \;\; and \;\; i\neq L_{G_i(g)}(g),\\ F_{M_g-1}(g)+1. \text{ if } \quad i=M_g. \end{cases} \tag{16}$$

Eq. (16) can be elaborated as follows:

1. When $i=0$, $F_0(g)=0$.
2. When $i\neq L_{G_i(g)}(g)$, the strength of one node at level i is $r^{G_{i+1}}+r^{G_i}$. With probability $\frac{r^{G_i}}{r^{G_{i+1}}+r^{G_i}}=\frac{1}{2}$, the walker takes one time step from a node at level i to a neighbor node at level $i-1$, and then makes $F_{i-1}(g)$ jumps to reach the trap for the first time; with probability $\frac{r^{G_{i+1}}}{r^{G_{i+1}}+r^{G_i}}=\frac{1}{2}$, the walker takes one time step from a node at level i to a neighbor node at level $i+1$ and then takes $F_{i+1}(g)$ steps to visit the trap for the first time.
3. When $i=L_{G_i(g)}(g)$, the strength of one node at level i is $r^{G_i-1}+(m-1)r^{G_i}$. With probability $\frac{r^{G_i-1}}{r^{G_i-1}+(m-1)r^{G_i}}=\frac{1}{(m-1)r+1}$, the particle, starting from a node at level i, walks one time step to arrive at a neighbor node at level $i-1$, and then jumps $F_{i-1}(g)$ to more steps to reach the trap for the first time; with probability $\frac{(m-1)r^{G_i}}{r^{G_i-1}+(m-1)r^{G_i}}=\frac{(m-1)r}{(m-1)r+1}$, the particle walks one time step from a node at level i to a neighbor node at level $i+1$ and then takes $F_{i+1}(g)$ steps to visit the trap for the first time.
4. When $i=M_g$, the walker only need to take time step from the outermost node to the second outermost node.

Thus, for $0<i<M_g$, we have

$$F_i(g)-F_{i-1}(g)=(m-1)r[F_{i+1}(g)-F_i(g)]+[1+(m-1)r], \quad i=L_{G_i(g)}(g), \tag{17}$$

and

$$F_i(g)-F_{i-1}(g)=[F_{i+1}(g)-F_i(g)]+2, \quad i\neq L_{G_i(g)}(g). \tag{18}$$

Let $A_i(g)=F_i(g)-F_{i-1}(g)$, then

$$A_i(g)=\begin{cases}(m-1)rA_{i+1}(g)+[1+(m-1)r], \text{ if } \quad i=L_{G_i(g)}(g),\\ A_{i+1}(g)+2. \text{ if } \quad i\neq L_{G_i(g)}(g),\end{cases} \tag{19}$$

holds for all $0<i<M_g$. Using the initial condition $A_{M_g}(g)=F_{M_g}(g)-F_{M_g-1}(g)=1$, Eq. (19) can be solved to yield

$$
\begin{aligned}
A_i(g) &= (mr-r)^{g-G_i(g)} + [1+(m-1)r][(mr-r)^0 + (mr-r)^1 + (mr-r)^2 + \cdots + (mr-r)^{g-1-G_i(g)}] \\
&+ 2[n^{g-1} - n^{g-1-G_i(g)} - i] + 2(mr-r)\{[(mr-r)^0 \cdot n^{g-2-G_i(g)} + (mr-r)^1 \cdot n^{g-3-G_i(g)} \\
&+ \cdots + (mr-r)^{g-3-G_i(g)} \cdot n^1] - [(mr-r)^0 + (mr-r)^1 + \cdots + (mr-r)^{g-3-G_i(g)})]\} \\
&= (mr-r)^{g-G_i(g)} + [1+(m-1)r] \sum_{j=0}^{g-1-G_i(g)} (mr-r)^i + 2[n^{g-1} - n^{g-1-G_i(g)} - i] \\
&+ 2(mr-r)[\sum_{j=0}^{g-3-G_i(g)} (mr-r)^i \cdot n^{g-2-G_i(g)-i} - \sum_{j=0}^{g-3-G_i(g)} (mr-r)^i].
\end{aligned}
$$

If $r \neq \frac{n}{m-1}$ and $r \neq \frac{1}{n(m-1)}$, then we have

$$
\begin{aligned}
A_i(g) &= 2[1 + \frac{1}{mr-r-1} + \frac{n}{(mr-r-n)(mr-r)} - \frac{1}{(mr-r-1)(mr-r)}](mr-r)^{g-G_i(g)} \\
&+ 2 \cdot n^{g-1} - \frac{4(mr-r)-2n}{mr-r-n} n^{g-1-G_i(g)} - 2i + 1.
\end{aligned}
$$

If $r = \frac{n}{m-1}$, then we have

$$A_i(g) = 2(g-1-G_i(g))n^{g-1-G_i(g)} + 2(n-1)n^{g-1-G_i(g)} + 2 \cdot n^{g-1} - 2i + 1.$$

If $r = \frac{1}{m-1}$, then we have

$$A_i(g) = 2 \cdot n^{g-1} - \frac{2(n-2)}{n-1} n^{g-1-G_i(g)} - 2i + \frac{3n-5}{n-1}.$$

Using the obtained intermediate quantity $A_i(g)$, $F_i(g)$ $(0 \leq i \leq M_g)$ can be calculated by

$$
\begin{aligned}
F_i(g) &= F_0(g) + \sum_{j=1}^{i} [F_j(g) - F_{j-1}(g)] \\
&= F_0(g) + \sum_{j=1}^{i} A_j(g).
\end{aligned}
\tag{20}
$$

When $r \neq \frac{n}{m-1}$, $r \neq \frac{1}{m-1}$ and $r \neq \frac{1}{n(m-1)}$, then

$$
\begin{aligned}
F_i(g) &= \frac{k_1}{n[n(mr-r)-1]}[n(mr-r)]^g + k_1 \cdot i(mr-r)^{g-G_i(g)} - \frac{k_1}{n(n-1)} n^g (mr-r)^{g-G_i(g)} \\
&+ \frac{k_1(mr-r-1)}{(n-1)[n(mr-r)-1]}[n(mr-r)]^{g-G_i(g)} + 2i \cdot n^{g-1} - \frac{k_2}{n^2-1} n^{2(g-1)} - k_2 \cdot i \cdot n^{g-1-G_i(g)} \\
&+ \frac{k_2}{n-1} n^{2g-2-G_i(g)} - \frac{k_2}{n(n^2-1)} n^{2(g-G_i(g))} - i^2,
\end{aligned}
\tag{21}
$$

where $k_1 = 2[1 + \frac{1}{mr-r-1} + \frac{n}{(mr-r-n)(mr-r)} - \frac{1}{(mr-r-1)(mr-r)}]$ and $k_2 = \frac{4(mr-r)-2n}{mr-r-n}$.

When $r = \frac{n}{m-1}$, then

$$\begin{aligned} F_i(g) &= \frac{2[(n^2-1)g+(n^3-3n^2-n+2)]}{(n^2-1)^2}n^{2(g-1)}+2i(g+n-2-G_i(g))n^{g-1-G_i(g)} \\ &+ \frac{2[(n^2-1)g-(n^2-1)G_i(g)+(n^3-2n^2+2)]}{n(n^2-1)^2}n^{2(g-G_i(g))}-\frac{2(g+n-2-G_i(g))}{n-1}n^{2g-2-G_i(g)} \\ &+ 2i\cdot n^{g-1}-i^2. \end{aligned} \tag{22}$$

When $r=\frac{1}{m-1}$, then

$$\begin{aligned} F_i(g) &= -2(n-2)[\frac{n^{2(g-1)}}{(n+1)(n-1)^2}+\frac{i\cdot n^{g-1-G_i(g)}}{n-1}+\frac{n^{2(g-G_i(g))}}{n(n+1)(n-1)^2} \\ &- \frac{n^{2g-2-G_i(g)}}{(n-1)^2}]+2i\cdot n^{g-1}-i^2+\frac{2(n-2)}{n-1}. \end{aligned} \tag{23}$$

When $r=\frac{1}{n(m-1)}$, then

$$\begin{aligned} F_i(g) &= \frac{2(n^2-n-1)}{n^2-1}[\frac{G_i(g)}{n}+\frac{1}{n(n-1)}+i\cdot n^{-g+G_i(g)}-\frac{n^{G_i(g)}}{n(n-1)}] \\ &- \frac{2(n^2-2)}{n^2-1}[\frac{n^{2g}}{n^2(n^2-1)}+\frac{n^{2(g-G_i(g))}}{n(n^2-1)}+\frac{i\cdot n^{g-G_i(g)}}{n}-\frac{n^{2g-G_i(g)}}{n^2(n-1)}]+2i\cdot n^{g-1}-i^2. \end{aligned} \tag{24}$$

$\langle T\rangle_g$ is the ATT to the trap located on the central node in C_g. According to Eqs. (15) and (21-24), $\langle T\rangle_g$ can be solved to obtain as follows:

$$\begin{aligned} \langle T\rangle_g &= \frac{\sum_{i=1}^{M_g}N_i(g)F_i(g)}{N_g-1} \\ &= \{\frac{k_1[n(mr-r)]^g}{n[n(mr-r)-1]}\frac{m}{m-1}[\sum_{i=1}^{g-1}(m-1)^i\cdot n^{g-1-i}+(m-1)^g]+\frac{k_1m(mr-r)^g}{m-1} \\ &\times \sum_{i=1}^{M_g}i\cdot r^{-G_i(g)}-\frac{k_1m[nr(m-1)]^g}{n(n-1)(m-1)}[\sum_{i=1}^{g-1}r^{-i}\cdot n^{g-1-i}+r^{-g}] \\ &+ \frac{k_1m(mr-r-1)[n(mr-r)]^g}{(m-1)(n-1)[n(mr-r)-1]}[\sum_{i=1}^{g-1}(nr)^{-i}\cdot n^{g-1-i}+(nr)^{-g}]+\frac{2m\cdot n^{g-1}}{m-1} \\ &\times \sum_{i=1}^{M_g}i\cdot(m-1)^{G_i(g)}-\frac{k_2m\cdot n^{2(g-1)}}{(m-1)(n^2-1)}[\sum_{i=1}^{g-1}(m-1)^i\cdot n^{g-1-i}+(m-1)^g] \\ &- \frac{k_2m\cdot n^{g-1}}{m-1}\sum_{i=1}^{M_g}i\cdot(\frac{m-1}{n})^{G_i(g)}+\frac{k_2m\cdot n^{2(g-1)}}{(m-1)(n-1)}[\sum_{i=1}^{g-1}(\frac{m-1}{n})^i\cdot n^{g-1-i} \\ &+ (\frac{m-1}{n})^g]-\frac{k_2m\cdot n^{2g-1}}{(m-1)(n^2-1)}[\sum_{i=1}^{g-1}(\frac{m-1}{n^2})^i\cdot n^{g-1-i}+(\frac{m-1}{n^2})^g] \\ &- \frac{m}{m-1}\sum_{i=1}^{M_g}i^2\cdot(m-1)^{G_i(g)}\}/(N_g-1), \end{aligned} \tag{25}$$

for $r\neq\frac{n}{m-1}$, $r\neq\frac{1}{m-1}$ and $r\neq\frac{1}{n(m-1)}$.

$$\begin{aligned}
\langle T\rangle_g &= \frac{\sum_{i=1}^{M_g} N_i(g)F_i(g)}{N_g-1}\\
&= \{\frac{2m[(n^2-1)g+(n^3-3n^2-n+2)]n^{2(g-1)}}{(m-1)(n^2-1)^2}[\sum_{i=1}^{g-1}(m-1)^i\cdot n^{g-1-i}+(m-1)^g]\\
&+ \frac{2m(g+n-2)n^{g-1}}{m-1}\sum_{i=1}^{M_g} i\cdot(\frac{m-1}{n})^{G_i(g)}-\frac{2m\cdot n^{g-1}}{m-1}\sum_{i=1}^{M_g} i\cdot G_i(g)(\frac{m-1}{n})^{G_i(g)}\\
&+ \frac{2m[(n^2-1)g+(n^3-2n^2+2)]n^{2g-1}}{(m-1)(n^2-1)^2}[\sum_{i=1}^{g-1}(\frac{m-1}{n^2})^i\cdot n^{g-1-i}+(\frac{m-1}{n^2})^g]-\frac{2m\cdot n^{2g-1}}{(m-1)(n^2-1)}\\
&\times \sum_{i=1}^{M_g} G_i(g)\cdot(\frac{m-1}{n^2})^{G_i(g)}-\frac{2m(g+n-2)n^{2(g-1)}}{(m-1)(n-1)}[\sum_{i=1}^{g-1}(\frac{m-1}{n})^i\cdot n^{g-1-i}+(\frac{m-1}{n})^g]\\
&+ \frac{2m\cdot n^{2(g-1)}}{(m-1)(n-1)}\sum_{i=1}^{M_g} G_i(g)\cdot(\frac{m-1}{n})^{G_i(g)}+\frac{2m\cdot n^{g-1}}{m-1}\sum_{i=1}^{M_g} i\cdot(m-1)^{G_i(g)}\\
&- \frac{m}{m-1}\sum_{i=1}^{M_g} i^2\cdot(m-1)^{G_i(g)}\}/(N_g-1), \qquad (26)
\end{aligned}$$

for $r=\frac{n}{m-1}$.

$$\begin{aligned}
\langle T\rangle_g &= \frac{\sum_{i=1}^{M_g} N_i(g)F_i(g)}{N_g-1}\\
&= \{\frac{-2m(n-2)n^{2(g-1)}}{(m-1)(n+1)(n-1)^2}[\sum_{i=1}^{g-1}(m-1)^i\cdot n^{g-1-i}+(m-1)^g]-\frac{2m(n-2)n^{g-1}}{(m-1)(n-1)}\\
&\times \sum_{i=1}^{M_g} i\cdot(\frac{m-1}{n})^{G_i(g)}-\frac{2m(n-2)n^{2g-1}}{(m-1)(n+1)(n-1)^2}[\sum_{i=1}^{g-1}(\frac{m-1}{n^2})^i\cdot n^{g-1-i}+(\frac{m-1}{n^2})^g]\\
&+ \frac{2m(n-2)n^{2(g-1)}}{(m-1)(n-1)^2}[\sum_{i=1}^{g-1}(\frac{m-1}{n})^i\cdot n^{g-1-i}+(\frac{m-1}{n})^g]+\frac{2m\cdot n^{g-1}}{m-1}\sum_{i=1}^{M_g} i\cdot(m-1)^{G_i(g)}\\
&- \frac{m}{m-1}\sum_{i=1}^{M_g} i^2\cdot(m-1)^{G_i(g)}+\frac{2m(n-2)}{(m-1)(n-1)}[\sum_{i=1}^{g-1}(m-1)^i\cdot n^{g-1-i}+(m-1)^g]\}/(N_g-1), \qquad (27)
\end{aligned}$$

for $r=\frac{1}{m-1}$.

$$\begin{aligned}
\langle T\rangle_g &= \frac{\sum_{i=1}^{M_g} N_i(g)F_i(g)}{N_g-1}\\
&= \{\frac{2(n^2-n-1)}{n^2-1}\langle\frac{m}{n(m-1)}[\sum_{i=1}^{g-1} i\cdot(m-1)^i\cdot n^{g-1-i}+g\cdot(m-1)^g]+\frac{m}{n(n-1)(m-1)}\\
&\times [\sum_{i=1}^{g-1}(m-1)^i\cdot n^{g-1-i}+(m-1)^g]+\frac{mn^{-g}}{m-1}\sum_{i=1}^{M_g} i\cdot(mn-n)^{G_i(g)}-\frac{m}{n(n-1)(m-1)}\\
&\times [n^{g-1}\sum_{i=1}^{g-1}(m-1)^i+(mn-n)^g]\rangle-\frac{2(n^2-2)}{n^2-1}\langle\frac{m\cdot n^{2g}}{n^2(n^2-1)(m-1)}[\sum_{i=1}^{g-1}(m-1)^i\cdot n^{g-1-i}
\end{aligned}$$

$$\begin{aligned}
+ & \quad (m-1)^g] + \frac{m \cdot n^{2g}}{n(n^2-1)(m-1)}[\sum_{i=1}^{g-1}(\frac{m-1}{n^2})^i \cdot n^{g-1-i} + (\frac{m-1}{n^2})^g] + \frac{m \cdot n^g}{n(m-1)}\sum_{i=1}^{M_g} i \cdot (\frac{m-1}{n})^{G_i(g)} \\
+ & \quad \frac{m \cdot n^{2g}}{n^2(n-1)(m-1)}[\sum_{i=1}^{g-1}(\frac{m-1}{n})^i \cdot n^{g-1-i} + (\frac{m-1}{n})^g]\rangle + \frac{2m \cdot n^{g-1}}{m-1}\sum_{i=1}^{M_g} i \cdot (m-1)^{G_i(g)} \\
- & \quad \frac{m}{m-1}\sum_{i=1}^{M_g} i^2 \cdot (m-1)^{G_i(g)}\}/(N_g-1),
\end{aligned} \tag{28}$$

for $r = \frac{1}{n(m-1)}$.

When $g \to \infty$, $\langle T \rangle_g$ has the following dominant term:

$$\langle T \rangle_g = \begin{cases} (n^2)^g, & \text{if} \quad r < \dfrac{n}{m-1}, \\ g \cdot (n^2)^g, & \text{if} \quad r = \dfrac{n}{m-1}, \\ [nr(m-1)]^g. & \text{if} \quad r > \dfrac{n}{m-1}. \end{cases} \tag{29}$$

In order to better analyze and understand the average trapping time (ATT), we have calculated the specific analytical expressions for ATTs under two scenarios:(1) m=5, n=2, and (2) m=5, n=4.

(1) When $m = 5$ and $n = 2$, we have

$$A_i(g) = \begin{cases} \dfrac{4r-1}{2r-1}[(4r)^{g-G_i(g)} - 2^{g-G_i(g)}] + 2^g - 2i + 1, & \text{if} \quad r \neq \dfrac{1}{2}, \\ \dfrac{g+1}{2}2^{g-G_i(g)} - \dfrac{G_i(g)}{2}2^{g-G_i(g)} + 2^g - 2i + 1, & \text{if} \quad r = \dfrac{1}{2}. \end{cases} \tag{30}$$

Then, $F_i(g)$ can be obtained as

$$F_i(g) = \begin{cases} \dfrac{4r-1}{2r-1}[\dfrac{(8r)^g}{2(8r-1)} + \dfrac{4r-1}{8r-1}(8r)^{g-G_i(g)} + i \cdot (4r)^{g-G_i(g)} - 2^{g-1}(4r)^{g-G_i(g)} - \dfrac{4^g}{6} \\ \qquad - \dfrac{4^{g-G_i(g)}}{3} - i \cdot 2^{g-G_i(g)} + 2^{2g-1-G_i(g)}] + i \cdot 2^g - i^2, \quad \text{if} \quad r \neq \dfrac{1}{2}, r \neq \dfrac{1}{8} \\ \dfrac{3g-1}{36}4^g + \dfrac{3g+5}{18}4^{g-G_i(g)} + \dfrac{g+1}{2}i \cdot 2^{g-G_i(g)} - \dfrac{g+1}{4}2^{2g-G_i(g)} - \dfrac{G_i(g)}{6}4^{g-G_i(g)} \\ \qquad - \dfrac{i}{2}G_i(g)2^{g-G_i(g)} + \dfrac{G_i(g)}{4}2^{2g-G_i(g)} + i \cdot 2^g - i^2, \quad \text{if} \quad r = \dfrac{1}{2}, \\ \dfrac{G_i(g)+1}{3} + \dfrac{2i}{3}2^{-g+G_i(g)} - \dfrac{2^{G_i(g)}}{3} - \dfrac{4^g}{9} - \dfrac{2 \cdot 4^{g-G_i(g)}}{9} - \dfrac{2i}{3}2^{g-G_i(g)} \\ \qquad + \dfrac{2^{2g-G_i(g)}}{3} + i \cdot 2^g - i^2, \quad \text{if} \quad r = \dfrac{1}{8}. \end{cases} \tag{31}$$

At last, $\langle T \rangle_g$ can be obtained as

$$\begin{aligned}
\langle T \rangle_g \quad = & \quad \{\frac{5(4r-1)}{8(2r-1)(8r-1)}[\frac{3}{2} \cdot (32r)^g - (16r)^g] + \frac{5(4r-1)}{4(2r-1)(8r-1)}[\frac{1}{2} \cdot (16r)^g + (2r-1)4^g] \\
+ & \quad [\frac{5(r-1)(4r-1)}{8(2r-1)^2}8^g + \frac{5r(r-1)}{4(2r-1)^2}4^g + \frac{5(4r-1)}{16(2r-1)^2}(8r)^g + \frac{5(2r+1)}{32(2r-1)^2}(16r)^g] \\
- & \quad \frac{5(4r-1)}{16(2r-1)^2}[2(r-1)8^g + (16r)^g] - \frac{5(4r-1)}{48(2r-1)}[3 \cdot 16^g - 2 \cdot 8^g] - \frac{5(4r-1)}{24(2r-1)}8^g
\end{aligned}$$

$$
\begin{aligned}
&- \frac{5(4r-1)}{32(2r-1)}[(2g-1)8^g - 2(g+2)4^g] + \frac{5(4r-1)}{16(2r-1)}(g+1)8^g + [\frac{15}{16}\cdot 16^g - \frac{5(3g-1)}{32}\cdot 8^g \\
&- \frac{5}{8}\cdot 4^g] - \frac{5}{4}[\frac{3}{8}\cdot 16^g - \frac{9g-16}{24}\cdot 8^g - \frac{3g+5}{8}\cdot 4^g - \frac{1}{6}\cdot 2^g]\}/(\frac{15}{2}\cdot 4^{g-1} - \frac{5}{2}\cdot 2^{g-1}),
\end{aligned}
$$

$$
\begin{aligned}
\langle T\rangle_g &= \{\frac{5(4r-1)}{8(2r-1)(8r-1)}[\frac{3}{2}\cdot(32r)^g - (16r)^g] + \frac{5(4r-1)}{4(2r-1)(8r-1)}[\frac{1}{2}\cdot(16r)^g + (2r-1)4^g] \\
&+ [\frac{5(r-1)(4r-1)}{8(2r-1)^2}8^g + \frac{5r(r-1)}{4(2r-1)^2}4^g + \frac{5(4r-1)}{16(2r-1)^2}(8r)^g + \frac{5(2r+1)}{32(2r-1)^2}(16r)^g] \\
&- \frac{5(4r-1)}{16(2r-1)^2}[2(r-1)8^g + (16r)^g] - \frac{5(4r-1)}{48(2r-1)}[3\cdot 16^g - 2\cdot 8^g] - \frac{5(4r-1)}{24(2r-1)}8^g \\
&- \frac{5(4r-1)}{32(2r-1)}[(2g-1)8^g - 2(g+2)4^g] + \frac{5(4r-1)}{16(2r-1)}(g+1)8^g + [\frac{15}{16}\cdot 16^g - \frac{5(3g-1)}{32}\cdot 8^g \\
&- \frac{5}{8}\cdot 4^g] - \frac{5}{4}[\frac{3}{8}\cdot 16^g - \frac{9g-16}{24}\cdot 8^g - \frac{3g+5}{8}\cdot 4^g - \frac{1}{6}\cdot 2^g]\}/(\frac{15}{2}\cdot 4^{g-1} - \frac{5}{2}\cdot 2^{g-1}),
\end{aligned}
$$

for $r \neq \frac{1}{2}$ and $r \neq \frac{1}{8}$.

$$
\begin{aligned}
\langle T\rangle_g &= \{\frac{5(3g-1)}{144}[\frac{3}{2}\cdot 16^g - 8^g] + \frac{5(3g+5)}{144}\cdot 8^g + \frac{5(g+1)}{64}[(2g-1)8^g + 2(g+2)4^g] \\
&- \frac{5(g+1)^2}{32}\cdot 8^g - \frac{5}{24}(8^g - 4^g) - \frac{5}{64}[(g^2+3g-6)8^g + (g^2+5g+6)4^g] + \frac{5(g^2+3g)}{64}\cdot 8^g \\
&+ [\frac{15}{16}\cdot 16^g - \frac{5(3g-1)}{32}\cdot 8^g - \frac{5}{8}\cdot 4^g] - \frac{5}{4}[\frac{3}{8}\cdot 16^g - \frac{9g-16}{24}\cdot 8^g - \frac{3g+5}{8}\cdot 4^g \\
&- \frac{1}{6}\cdot 2^g]\}/(\frac{15}{2}\cdot 4^{g-1} - \frac{5}{2}\cdot 2^{g-1}),
\end{aligned}
$$

for $r = \frac{1}{2}$.

$$
\begin{aligned}
\langle T\rangle_g &= \{\frac{5(3g+1)4^g}{24} + \frac{5}{144}(14\cdot 8^g - 7\cdot 4^g + 10\cdot 2^g - 8) - \frac{5}{72}(7\cdot 8^g - 2^{g+2}) \\
&- \frac{5}{36}(\frac{3}{2}\cdot 16^g - 8^g) - \frac{5}{36}\cdot 8^g - \frac{5}{48}[(2g-1)8^g + 2(g+2)4^g] + \frac{5(g+1)8^g}{24} \\
&+ [\frac{15}{16}\cdot 16^g - \frac{5(3g-1)}{32}\cdot 8^g - \frac{5}{8}\cdot 4^g] - \frac{5}{4}[\frac{3}{8}\cdot 16^g - \frac{9g-16}{24}\cdot 8^g - \frac{3g+5}{8}\cdot 4^g \\
&- \frac{1}{6}\cdot 2^g]\}/(\frac{15}{2}\cdot 4^{g-1} - \frac{5}{2}\cdot 2^{g-1}),
\end{aligned} \tag{32}
$$

for $r = \frac{1}{8}$.

(2) When $m = 5$ and $n = 4$, we have

$$
A_i(g) = \begin{cases} \frac{4r-3}{2(r-1)}(4r)^{g-G_i(g)} - \frac{2r-1}{2(r-1)}4^{g-G_i(g)} + \frac{1}{2}\cdot 4^g - 2i + 1, & \text{if} \quad r \neq 1, \\ \frac{g+2}{2}4^{g-G_i(g)} - \frac{G_i(g)}{2}4^{g-G_i(g)} + \frac{1}{2}\cdot 4^g - 2i + 1, & \text{if} \quad r = 1. \end{cases} \tag{33}
$$

Then, $F_i(g)$ can be obtained as

$$F_i(g)=\begin{cases}\frac{4r-3}{2(r-1)}[\frac{(16r)^g}{4(16r-1)}+\frac{4r-1}{3(16r-1)}(16r)^{g-G_i(g)}+i\cdot(4r)^{g-G_i(g)}-\frac{4^g}{12}\cdot(4r)^{g-G_i(g)}] \\ \quad -\frac{2r-1}{2(r-1)}[\frac{16^g}{60}+\frac{16^{g-G_i(g)}}{15}+i\cdot 4^{g-G_i(g)}-\frac{4^{2g-G_i(g)}}{12}]+\frac{i}{2}\cdot 4^g-i^2, \text{ if } r\neq 1, r\neq\frac{1}{16} \\ \frac{15g+14}{8\cdot 15^2}16^g+\frac{15g+34}{2\cdot 15^2}16^{g-G_i(g)}+\frac{g+2}{2}i\cdot 4^{g-G_i(g)}-\frac{g+2}{24}4^{2g-G_i(g)}-\frac{G_i(g)}{30}16^{g-G_i(g)} \\ \quad -\frac{i}{2}G_i(g)4^{g-G_i(g)}+\frac{G_i(g)}{24}4^{2g-G_i(g)}+\frac{i}{2}\cdot 4^g-i^2, \text{ if } r=1, \\ \frac{22}{15}[\frac{G_i(g)}{4}+\frac{1}{12}+i\cdot 4^{-g+G_i(g)}-\frac{4^{G_i(g)}}{12}]-\frac{7}{15}[\frac{16^g}{60}+\frac{16^{g-G_i(g)}}{15}+i\cdot 4^{g-G_i(g)} \\ \quad -\frac{4^{2g-G_i(g)}}{12}]+\frac{i}{2}\cdot 4^g-i^2, \text{ if } r=\frac{1}{16}.\end{cases} \tag{34}$$

At last, $\langle T\rangle_g$ can be obtained as

$$\begin{aligned}\langle T\rangle_g &= \{\frac{5(4r-3)}{128(r-1)(16r-1)}(g+3)(64r)^g+\frac{5(4r-1)(4r-3)}{24(r-1)(16r-1)^2}[\frac{1}{4}\cdot(64r)^g+(12r-1)4^g] \\ &+ \frac{5(4r-3)}{8(r-1)}[\frac{9(4r+1)}{32(4r-1)(16r-1)}(64r)^g+\frac{3}{8(4r-1)}(16r)^g+\frac{r-1}{4(4r-1)}\cdot 16^g+\frac{6r(r-1)}{(4r-1)(16r-1)}\cdot 4^g] \\ &- \frac{5(4r-3)}{96(r-1)(4r-1)}[\frac{1}{4}\cdot(64r)^g+(3r-1)\cdot 16^g]-\frac{(2r-1)(g+3)}{6\cdot 4^3(r-1)}64^g-\frac{2r-1}{24\cdot 60(r-1)}[64^g+44\cdot 4^g] \\ &- \frac{5(2r-1)}{256(r-1)}[64^g+4\cdot 16^g]+\frac{5(2r-1)}{6\cdot 4^3(r-1)}64^g+[\frac{15(2g-1)64^g}{256}+\frac{5(3g-2)16^g}{64}]-\frac{1}{256}[3(5g-11)64^g \\ &+ 10(6g-1)16^g+8(5g+6)4^g]\}/(5(g+3)4^{g-2})\end{aligned}$$

for $r\neq 1$, $r\neq\frac{1}{4}$ and $r\neq\frac{1}{16}$.

At last, $\langle T\rangle_g$ can be obtained as

$$\begin{aligned}\langle T\rangle_g &= \frac{(15g+14)(g+3)64^g}{90\cdot 4^3}+\frac{15g+34}{24\cdot 15^2}[64^g+44\cdot 4^g]+\frac{5(g+2)}{256}[64^g+4\cdot 4^g] \\ &- \frac{5(g+2)}{6\cdot 4^3}64^g-\frac{1}{24*15^2}[4\cdot 64^g+(165g-4)4^g]-\frac{1}{96}[3\cdot 64^g+5\cdot 16^g-8\cdot 4^g] \\ &+ \frac{5}{27\cdot 32}[64^g+(6g-1)16^g]+[\frac{15(2g-1)64^g}{256}+\frac{5(3g-2)16^g}{64}]-\frac{1}{256}[3(5g-11)64^g \\ &+ 10(6g-1)16^g+8(5g+6)4^g]\}/(5(g+3)4^{g-2})\end{aligned}$$

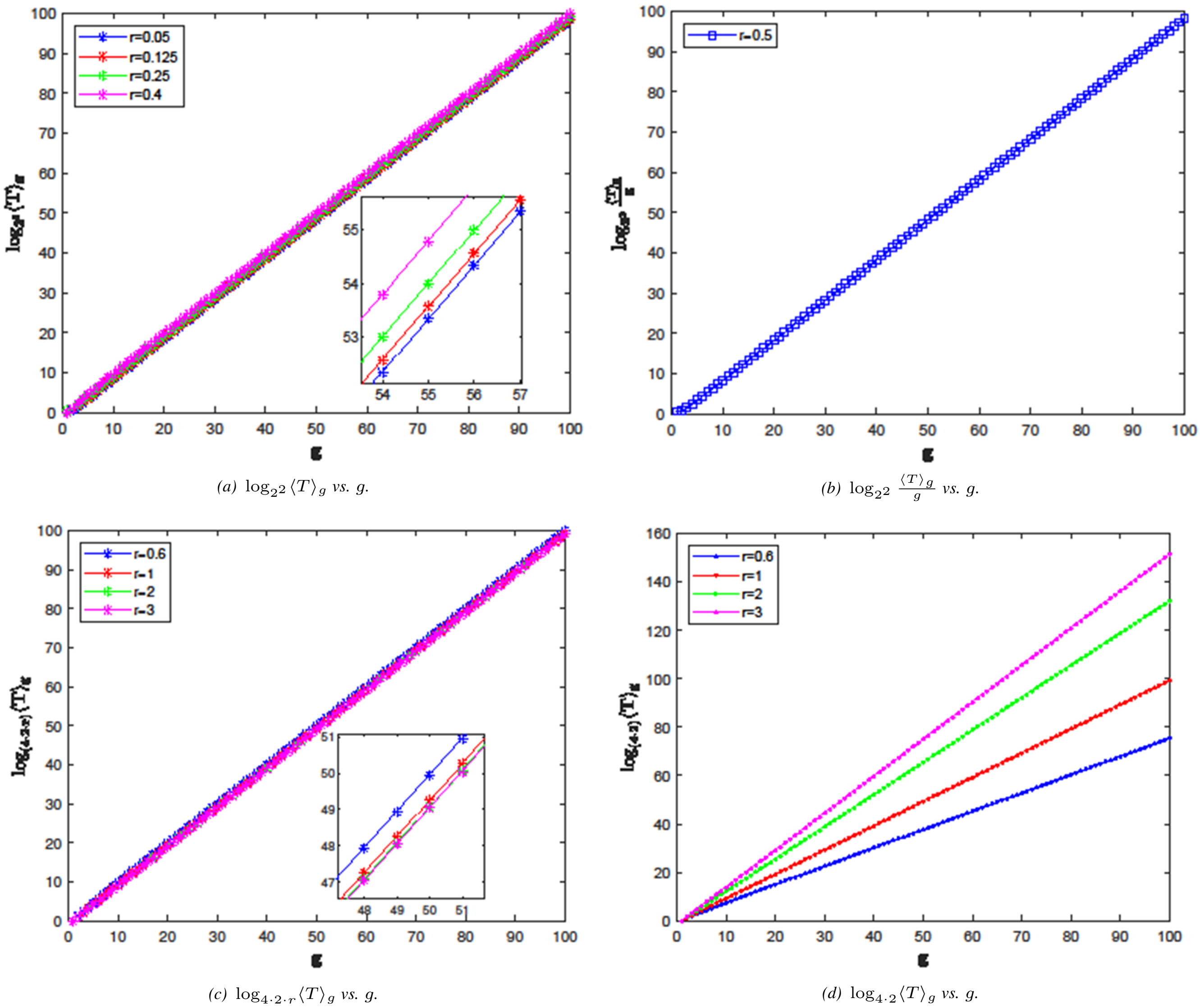
for $r=1$.

$$\begin{aligned}\langle T\rangle_g &= \{-\frac{(g+3)64^g}{9\cdot 64}-\frac{1}{15\cdot 12^2}[64^g+44\cdot 4^g]-\frac{5}{6\cdot 4^3}[64^g+4\cdot 16^g]+\frac{5\cdot 64^g}{4\cdot 12^2} \\ &+ [\frac{5(2g-1)}{32}16^g+\frac{5(3g+2)}{24}4^g]+[\frac{15(2g-1)64^g}{256}+\frac{5(3g-2)16^g}{64}]-\frac{1}{256}[3(5g-11)64^g \\ &+ 10(6g-1)16^g+8(5g+6)4^g]\}/(5(g+3)4^{g-2})\end{aligned}$$

for $r=\frac{1}{4}$.

$$\begin{aligned}\langle T\rangle_g &= \{\frac{11(g^2+7g)4^g}{8\cdot 24}+\frac{11(g+3)4^g}{12\cdot 24}+\frac{55}{192}[2\cdot 16^g-11(5g-11)4^g-176]-\frac{11}{6\cdot 12^2}[13\cdot 16^g-4^{g+1}]\\ &- \frac{7(g+3)64^g}{45\cdot 64}-\frac{7}{48\cdot 15^2}[64^g+44\cdot 4^g]-\frac{7}{12\cdot 32}[64^g+4\cdot 16^g]+\frac{7\cdot 64^g}{9\cdot 64}+[\frac{15(2g-1)64^g}{256}\\ &+ \frac{5(3g-2)16^g}{64}]-\frac{1}{256}[3(5g-11)64^g+10(6g-1)16^g+8(5g+6)4^g]\}/(5(g+3)4^{g-2})\end{aligned} \tag{35}$$

for $r=\frac{1}{16}$.



(a) $\log_{2^2}\langle T\rangle_g$ *vs. g.*

(b) $\log_{2^2}\frac{\langle T\rangle_g}{g}$ *vs. g.*

(c) $\log_{4\cdot 2\cdot r}\langle T\rangle_g$ *vs. g.*

(d) $\log_{4\cdot 2}\langle T\rangle_g$ *vs. g.*

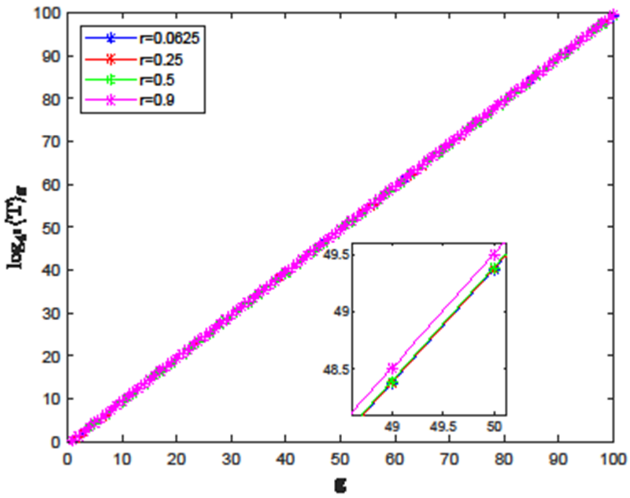
Figure 7. *Average trapping time $\langle T\rangle_g$ or $\frac{\langle T\rangle_g}{g}$ versus g on a semilogarithmic scale when $m=5$ and $n=2$.*

For the range of $g\leq 100$, the average trapping time $\langle T\rangle_g$ or $\frac{\langle T\rangle_g}{g}$ versus g on a semilogarithmic scale is shown in Figures 7 and 8. Regardless of the relationship between m and n, the dominant terms of ATTs are consistent according to Eq. (29).

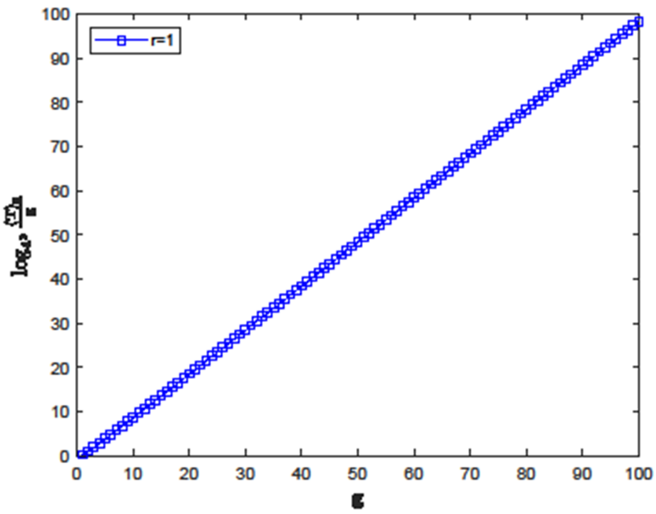
1. When $r<\frac{n}{m-1}$, the main term of ATT is the g power of n^2 according to Eq. (29). $\langle T\rangle_g$ versus g on a semilogarithmic scale is basically a straight line, and the slope of the fitted straight line is expressed as $k\approx 1$ in Figures 7(a) and 8(a), which is consistent with Eq. (29).
2. When $r=\frac{n}{m-1}$, the dominant term of $\frac{\langle T\rangle_g}{g}$ is the g power of n^2 based on Eq. (29). $\frac{\langle T\rangle_g}{g}$ versus g on a semilogarithmic scale is basically a straight line, and the slope of the fitted line is approximately 1 in Figures 7(b) and 8(b), which is consistent with Eq. (29). Therefore,

ATTs are less affected by the structural parameter m and the weight factor r when $r \leq \frac{n}{m-1}$, indicating that the efficiency of the trapping process is independent of m and r in Figures 7(a), 7(b), 8(a) and 8(b).

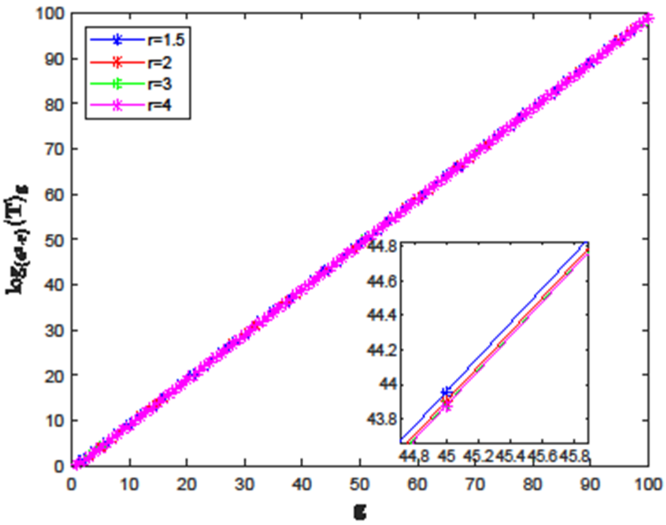
3. When $r > \frac{n}{m-1}$, the main term of $\langle T\rangle_g$ is the g power of $nr(m-1)$ based on Eq. (29), Figures 7(c) and 8(c). m and n are two key parameters for constructing the internal structure of the networks. Figures 7(d) and 8(d) indicate that ATTs grow sublinearly with the network order, which also means that the efficiency of the trapping process depends on the parameters m, n, and r. When m and n are kept fixed, the smaller the value of r is, the more efficient the trapping process is. When r is kept fixed, the smaller the value of m or n is, the more efficient the trapping process is. Therefore, the trapping efficiency of the weighted extended Cayley networks is not only affected by the underlying structures of the networks m and n, but also by the weight factor r.



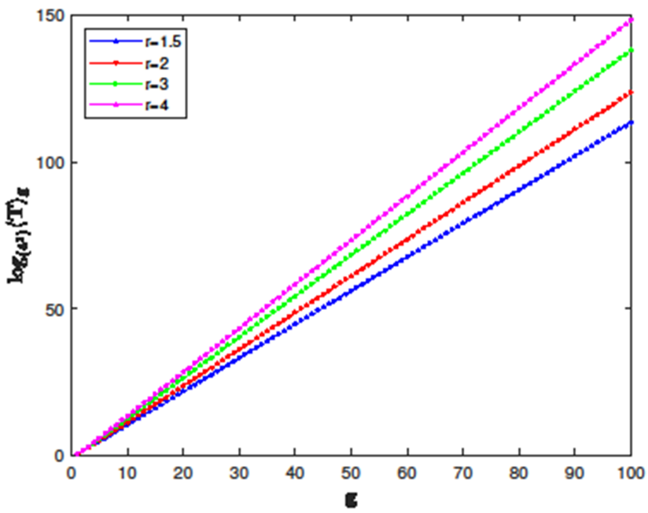
(a) $\log_{4^2}\langle T\rangle_g$ *vs. g.*



(b) $\log_{4^2}\frac{\langle T\rangle_g}{g}$ *vs. g.*



(c) $\log_{4^2\cdot r}\langle T\rangle_g$ *vs. g.*



(d) $\log_{4^2}\langle T\rangle_g$ *vs. g.*

Figure 8. *Average trapping time* $\langle T\rangle_g$ *or* $\frac{\langle T\rangle_g}{g}$ *versus g on a semilogarithmic scale when* $m = 5$ *and* $n = 4$.

5. Conclusions

In this paper, the weighted extended Cayley networks are constructed depending on two structural parameters of the network m, n and a weight factor r. We have used a new calculation method to calculate the exact analytic formula of the average weighted shortest path (AWSP). Firstly, the weighted extended Cayley network $C_{g+1}(m,n)$ should be split into $m+1$ groups, including $C_g^{(0)}(m,n), C_g^{(1)}(m,n), \cdots, C_g^{(m)}(m,n)$. Then, the sum of all the AWSPs with nodes which are not in the same branch of $C_{g+1}(m,n)$ in Appendix is obtained. The expressions of the

AWSPs with nodes belonging to $C_g^{(0)}(m,n)$ and $C_g^{(1)}(m,n)$ have also been deduced in Appendix. The expressions for AWSPs show that: (1) For $0 < r < n$, it grows with the g power of n. (2) For $r = n$, its dominant term is $g \times n^g$. (3) For $r > n$, AWSP $\times g$ grows as a power-law function of the weight factor r with the exponent, represented by the network iteration g when $n = m-1$; AWSP grows as a power-law function of the weight factor r with an exponent of g when $n \neq m-1$. For very large systems, the AWSPs for different value of weight factor r are less affected by the parameter m. Thus, the AWSPs are less affected by the weight factor r when $0 < r \leq n$, while the AWSPs depend on the weight factor r when $r > n$.

To explore the effect of the underlying structures and the weight factor on the trapping efficiency, we have studied the trapping problem for weighted-dependent walks taking place on a weighted extended Cayley network, concentrating on a particular case with the single trap fixed at the central node. Then, the exact analytic formula of the average trapping time (ATT) is derived based on the layering of weighted extended Cayley network. It was surprisingly found that (1) The main term of ATT is the g power of n^2 when $r < \frac{n}{m-1}$. (2) The dominant term of $\frac{\langle T\rangle_g}{g}$ is the g power of n^2. (3) The main term of $\langle T\rangle_g$ is the g power of $nr(m-1)$ when $r > \frac{n}{m-1}$. Regardless of the relationship between m and n, the dominant terms of ATTs are consistent. ATTs are less affected by the structural parameter m and the weight factor r when $r \leq \frac{n}{m-1}$, indicating that the efficiency of the trapping process is independent of m and r. When $r > \frac{n}{m-1}$, the efficiency of the trapping process depends on three main parameters: two structural parameters of the network m, n and a weight factor r, which means that the smaller the value of $nr(m-1)$ is, the more efficient the trapping process is. Therefore, the trapping efficiency of the weighted extended Cayley networks is not only affected by the underlying structures of the networks m and n, but also by the weight factor r.

Acknowledgments

None.

Appendix

Appendix I. Calculation Process of $\sum_{i,j\in C_g^{(0)}(m,n)} d_{ij}$

Making $S_g(m,n) = \sum_{i,j\in C_g^{(0)}(m,n)} d_{ij}$, we have

$$S_g(m,1) = S_{g-1}(m,1) + \frac{3m^2-2m}{2}g^2 - \frac{m^2-2m}{2}g. \tag{36}$$

Using the initial condition $S_1(m,1) = m^2$, Eq. (36) can be solved to yield

$$\begin{aligned} S_g(m,0) &= S_1(m,0) + \frac{3m^2-2m}{2}(1^2+2^2+\cdots+g^2) - \frac{m^2-2m}{2}(1+2+\cdots+g) - \frac{3m^2-2m}{2} + \frac{m^2-2m}{2}, \\ &= \frac{3m^2-2m}{2}(1^2+2^2+\cdots+g^2) - \frac{m^2-2m}{2}(1+2+\cdots+g), \\ &= \frac{g(g+1)[(3m^2-2m)g+2m]}{6}. \end{aligned} \tag{37}$$

By replacing g with n^{g-1} in Eq. (37), we can obtain

$$S_g(m,n) = \frac{(3m^2-2m)n^{3g-3} + 3m^2\cdot n^{2g-2} + 2m\cdot n^{g-1}}{6} \tag{38}$$

Appendix II. Calculation Process of Ω_g

Let Ω_g^{ij} denote the sum of all shortest paths with nodes in $\Omega_g^{(i)}$ and $\Omega_g^{(j)}(i,j = 0,1,2,\cdots,m)$. Using the structure of $C_g(m,n)$, the analytical expression for Ω_g can be rewritten as

$$\begin{aligned} \Omega_g &= \Omega_g^{10} + \Omega_g^{20} + \cdots + \Omega_g^{m0} + \Omega_g^{12} + \Omega_g^{13} + \cdots + \Omega_g^{1m}, \\ &+ \Omega_g^{23} + \Omega_g^{24} + \cdots + \Omega_g^{2m} + \cdots + \Omega_g^{(m-2)(m-1)} + \Omega_g^{(m-2)m} + \Omega_g^{(m-1)m} \\ &= m\Omega_g^{10} + \frac{m(m-1)}{2}\Omega_g^{12}. \end{aligned} \tag{39}$$

Due to the symmetry of network structure, $\Omega_g^{10} = \Omega_g^{20} = \cdots = \Omega_g^{30}, \Omega_g^{12} = \Omega_g^{13} = \cdots = \Omega_g^{1m} = \Omega_g^{23} = \Omega_g^{24} = \cdots = \Omega_g^{2m} = \cdots = \Omega_g^{(m-2)(m-1)} = \Omega_g^{(m-2)m} = \Omega_g^{(m-1)m}$.

In order to calculate Ω_g^{10} and Ω_g^{12}, we introduce the intermediate quantity Λ_g, where Λ_g represents the sum of $d_{i0}(g)$ starting off from the node $i(i \in C_g(m,n), i \neq 0)$ to the central node 0. By definition, we have

$$\Lambda_g = \sum_{i \in C_g(m,n), i \neq 0} d_{i0}.$$

Considering the self-similar network structure, Λ_g can evolve recursively as

$$\begin{aligned}
\Lambda_g &= \frac{m-1}{m} r\Lambda_{g-1} \times m + [(N_{g-1}-1) \times \frac{m-1}{m} \times n^{g-2}] \times m + [n^{g-2} + (n^{g-2}-1) + \cdots + 2 + 1)] \times m \\
&= (m-1)r\Lambda_{g-1} + (m-1)(N_{g-1}-1)n^{g-2} + m \cdot \frac{n^{2g-4} + n^{g-2}}{2}.
\end{aligned}$$

Then, we have

$$\Lambda_g = \begin{cases} nr\Lambda_{g-1} + (n+1)g \cdot n^{2g-4} + \dfrac{(n+1)(2n-3)}{2} n^{2g-4} + \dfrac{n+1}{2} n^{g-2} & , \text{ if } \quad n = m-1, \\ (m-1)r\Lambda_{g-1} + \dfrac{m(m-1)(m-n)}{m-1-n}(mn-n)^{g-2} - \dfrac{m(m+n-1)}{2(m-1-n)} n^{2(g-2)} + \dfrac{m}{2} n^{g-2}, & \text{ if } \quad n \neq m-1, \end{cases} \tag{40}$$

To reduce computational complexity, we chose the maximum and minimum values of n. The maximum and minimum values of n are $m-1$ and 2, respectively. When $m = 3$, $n = 0$ and $r = 1$, this special network is consistent with the extended dendrimers[63]. When $n = 1$ and $r = 1$, the network is the Cayley trees[43]. So the minimum value of n is chosen as 2. When $n = m-1$, Λ_g is inductively to obtain as follows:

$$\begin{aligned}
\Lambda_g &= (nr)^{g-1}\Lambda_1 + \frac{n+1}{n^4} \sum_{i=0}^{g-2} (g-i)(n^2)^{g-i}(nr)^i + \frac{(n+1)(2n-3)}{2n^4} \sum_{i=0}^{g-2} (n^2)^{g-i}(nr)^i, + \frac{n+1}{2n^2} \sum_{i=0}^{g-2} n^{g-i}(nr)^i, \\
&= (n+1)(nr)^{g-1} + \frac{n+1}{n^4} \sum_{i=0}^{g-2} (g-i)(n^2)^{g-i}(nr)^i + \frac{(n+1)(2n-3)}{2n^4} \sum_{i=0}^{g-2} (n^2)^{g-i}(nr)^i, + \frac{n+1}{2n^2} \sum_{i=0}^{g-2} n^{g-i}(nr)^i,
\end{aligned}$$

where the initial conditions $\Lambda_1 = \sum_{i=1}^{n+1} d_{i0} = n+1$.

When $n \neq m-1$, Λ_g is inductively to obtain as follows:

$$\begin{aligned}
\Lambda_g &= (mr-r)^{g-1}\Lambda_1 + \frac{m(m-1)(m-n)}{m-1-n} \sum_{i=0}^{g-2} (mn-n)^{g-2-i}(mr-r)^i - \frac{m(m+n-1)}{2(m-1-n)} \\
&\times \sum_{i=0}^{g-2} (n^2)^{g-2-i}(mr-r)^i + \frac{m}{2} \sum_{i=0}^{g-2} n^{g-2-i}(mr-r)^i, \\
&= m(mr-r)^{g-1} + \frac{m(m-1)(m-n)}{m-1-n} \sum_{i=0}^{g-2} (mn-n)^{g-2-i}(mr-r)^i - \frac{m(m+n-1)}{2(m-1-n)} \\
&\times \sum_{i=0}^{g-2} (n^2)^{g-2-i}(mr-r)^i + \frac{m}{2} \sum_{i=0}^{g-2} n^{g-2-i}(mr-r)^i,
\end{aligned}$$

where the initial conditions $\Lambda_1 = \sum_{i=1}^{m} d_{i0} = m$.

Using the calculated results of Λ_g, we have

$$
\begin{aligned}
\Omega_g^{10} &= \sum_{\substack{i\in C_g^{(1)}(m,n)\\ j\in C_g^{(0)}(m,n)}} d_{ij}\\
&= \sum_{\substack{i\in C_g^{(1)}(m,n)\\ j\in C_g^{(0)}(m,n)}} (d_{i1}+d_{1j})\\
&= (m\cdot n^{g-1}+1)\sum_{i\in C_g^{(1)}(m,n)} d_{i1}+\frac{(m-1)(N_g-1)}{m}\sum_{j\in C_g^{(0)}(m,n)} d_{j1}\\
&= (m\cdot n^{g-1}+1)\frac{(m-1)r\Lambda_g}{m}+\frac{(m-1)(N_g-1)}{m}\sum_{j\in C_g^{(0)}(m,n)} d_{j1}, \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
\Omega_g^{12} &= \sum_{\substack{i\in C_g^{(1)}(m,n)\\ j\in C_g^{(2)}(m,n)}} d_{ij}\\
&= \sum_{\substack{i\in C_g^{(1)}(m,n)\\ j\in C_g^{(2)}(m,n)}} (d_{i1}+d_{12}+d_{j2})\\
&= \frac{(m-1)(N_g-1)}{m}\sum_{i\in C_g^{(1)}(m,n)} d_{i1}+\frac{(m-1)^2(N_g-1)^2}{m^2}2\cdot n^{g-1}+\frac{(m-1)(N_g-1)}{m}\sum_{j\in C_g^{(2)}(m,n)} d_{j2}\\
&= \frac{2(m-1)(N_g-1)}{m}\frac{(m-1)r\Lambda_g}{m}+\frac{(m-1)^2(N_g-1)^2}{m^2}2\cdot n^{g-1}, \qquad (42)
\end{aligned}
$$

where $d_{12}=2\cdot n^{g-1}$ have been used. In order to Ω_g^{10}, we need to deduce the result of $\sum_{j\in C_g^{(0)}(m,n)} d_{j1}$. Then, we have

$$
\begin{aligned}
\sum_{j\in C_g^{(0)}(m,0)} d_{j1} &= g\sum_{j\in C_1^{(0)}(m,0)} d_{j1}+[1+(2g-1)\times(m-1)]+[2+(2g-2)\times(m-1)]+\cdots\\
&+ [(g-1)+(2g-(g-1))\times(m-1)]\\
&= (2m-1)g+\frac{g(g-1)}{2}+(m-1)(g-1)\cdot 2g-(m-1)\cdot\frac{g(g-1)}{2}\\
&= \frac{g[(3m-2)g+m]}{2}, \qquad (43)
\end{aligned}
$$

where $\sum_{j\in C_1^{(0)}(m,0)} d_{j1}=2m-1$.

By replacing g with n^{g-1} in Eq. (43), we can obtain

$$
\sum_{j\in C_g^{(0)}(m,n)} d_{j1}=\frac{(3m-2)n^{2g-2}+m\cdot n^{g-1}}{2}. \qquad (44)
$$

Substituting Eqs. (41), (42) and (44) into Eq. (39), we have

$$
\begin{aligned}
\Omega_g &= [m\cdot n^{g-1}+1+\frac{(m-1)^2(N_g-1)}{m}]\cdot(mr-r)\Lambda_g+\frac{(m-1)^3(N_g-1)^2}{m}n^{g-1}\\
&+ (m-1)(N_g-1)\frac{(3m-2)n^{2g-2}+m\cdot n^{g-1}}{2}.
\end{aligned}
$$

Appendix III. Calculation Process of $\sum_{i,j\in C_g^{(1)}(m,n)} d_{ij}$

In order to calculating $\sum_{i,j\in C_g^{(1)}(m,n)} d_{ij}$, we first divide the network structure $C_g^{(1)}(m,n)$ into m branches, including $B_g^{(0)}, B_g^{(1)}, \cdots, B_g^{(m-1)}$ (see figure 9). $B_g^{(1)}, B_g^{(2)}, \cdots, B_g^{(m-1)}$ are the copies of B_{g-1}, whose weighted edges have been scaled by the weight factor r. The white nodes do not belong to B_g or $D_g^{(1)}(m,n)$; the green nodes belong to $B_g^{(0)}$; and the red nodes belong to $B_g^{(1)} + B_g^{(2)} + \cdots + B_g^{(m-1)}$.

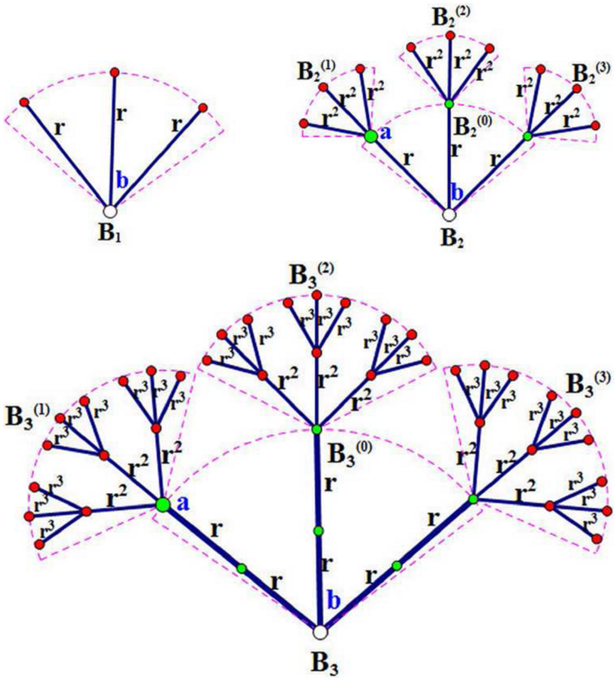


Figure 9. The structure of $C_g^{(1)}(m,n)$ *is split into* m *branches, including* $B_g^{(0)}, B_g^{(1)}, \cdots, B_g^{(m-1)}$.

Based on the network division of B_g, $\sum_{i,j\in B_g} d_{ij}$ can be rewritten as

$$\begin{aligned}\sum_{i,j\in B_g} d_{ij} &= \sum_{i,j\in B_g^{(0)}} d_{ij} + \sum_{i,j\in B_g^{(1)}} d_{ij} + \sum_{i,j\in B_g^{(2)}} d_{ij} + \cdots + \sum_{i,j\in B_g^{(m-1)}} d_{ij} + \sum_{i\in B_g^{(0)},j\in B_g^{(1)}} d_{ij} \\ &+ \cdots + \sum_{i\in B_g^{(0)},j\in B_g^{(m-1)}} d_{ij} + \sum_{i\in B_g^{(1)},j\in B_g^{(2)}} d_{ij} + \cdots + \sum_{i\in B_g^{(1)},j\in B_g^{(m-1)}} d_{ij} + \sum_{i\in B_g^{(2)},j\in B_g^{(3)}} d_{ij}, \\ &+ \cdots + \sum_{i\in B_g^{(2)},j\in B_g^{(m-1)}} d_{ij} + \cdots + \sum_{i\in B_g^{(m-2)},j\in B_g^{(m-1)}} d_{ij}, \\ &= \sum_{i,j\in B_g^{(0)}} d_{ij} + (m-1)\sum_{i,j\in B_g^{(1)}} d_{ij} + (m-1)\sum_{i\in B_g^{(0)},j\in B_g^{(1)}} d_{ij} + \frac{(m-1)(m-2)}{2}\sum_{i\in B_g^{(1)},j\in B_g^{(2)}} d_{ij},\end{aligned}$$

where $\sum_{i,j\in C_g^{(1)}(m,n)} d_{ij} = \sum_{i,j\in B_g} d_{ij}$.

Based on the structure of $B_g^{(0)}(m,0)$, we can obtain the iterative formula as follows

$$\begin{aligned}\sum_{i,j\in B_g^{(0)}(m,0)} d_{ij} - \sum_{i,j\in B_{g-1}^{(0)}(m,0)} d_{ij} &= r\frac{(m-1)(g-1)[(3m-5)g-(4m-6)]}{2} \\ &= \frac{(m-1)(3m-5)r}{2}(g-1)^2 - \frac{(m-1)^2 r}{2}(g-1).\end{aligned}$$

Then, we can get

$$\begin{aligned}\sum_{i,j\in B_g^{(0)}(m,0)} d_{ij} &= \frac{(m-1)(3m-5)r}{2}\cdot\sum_{i=1}^{g-1} i^2-\frac{(m-1)^2r}{2}\cdot\sum_{i=1}^{g-1} i+\frac{(m-1)^2r}{2}-\frac{(m-1)(3m-5)r}{2}+\sum_{i,j\in B_2^{(0)}(m,0)} d_{ij}\\ &= \frac{(m-1)(3m-5)r}{2}\sum_{i=1}^{g-1} i^2-\frac{(m-1)^2r}{2}\sum_{i=1}^{g-1} i\\ &= \frac{(m-1)gr(g-1)[(3m-5)g-(3m-4)]}{6}, \qquad (45)\end{aligned}$$

where $\sum_{i,j\in B_2^{(0)}(m,0)} d_{ij}=2C_{m-1}^2\cdot r=(m-1)(m-2)r$.

By replacing $g-1$ with n^{g-2} in Eq. (45), we can obtain

$$\sum_{i,j\in B_g^{(0)}(m,n)} d_{ij}=\frac{(m-1)n^{g-2}r(n^{g-2}+1)[(3m-5)n^{g-2}-1]}{6}. \qquad (46)$$

Let $\triangle_g=\sum_{i\in B_g^{(0)},j\in B_g^{(1)}} d_{ij}$ denote the sum of all shortest paths with nodes in $B_g^{(0)}$ and $B_g^{(1)}$. $\nabla_g=\sum_{i\in B_g^{(1)},j\in B_g^{(2)}} d_{ij}$ is expressed as the sum of all shortest paths with nodes in $B_g^{(1)}$ and $B_g^{(2)}$. Two intermediate quantities $\top_g$ and $\perp_g$ are introduced to calculate $\triangle_g$ and ∇_g, where $\top_g=\sum_{i\in B_g^{(0)}} d_{ia}$ and $\perp_g^{'}=\sum_{i\in B_g} d_{ib}$.

$$\begin{aligned}\top_g(m,0) &= (m-2)r[g+(g+1)+\cdots+(2g-2)]+r[1+2+\cdots+(g-2)]\\ &= \frac{r(g-1)[(3m-5)g-2(m-1)]}{2}. \qquad (47)\end{aligned}$$

By replacing $g-1$ with n^{g-2} in Eq. (47), we have

$$\top_g(m,n)=\frac{r\cdot n^{g-2}[(3m-5)n^{g-2}+(m-3)]}{2}.$$

The iteration formula of $\perp_g$ is written as follows:

$$\perp_g^{'}=\begin{cases} nr\perp_{g-1}^{'}+\dfrac{(2g+2n-3)r}{2}n^{2g-3}+\dfrac{r}{2}n^{g-1}, & n=m-1,\\ (mr-r)\perp_{g-1}^{'}+(mr-r)[\dfrac{(m-n)(m-1)}{m-n-1}(mn-n)^{g-2}-\dfrac{m+n-1}{2(m-n-1)}n^{2g-4}+\dfrac{1}{2}n^{g-2}], & n\neq m-1.\end{cases}$$

When $n=m-1$, $\perp_g^{'}$ is inductively to obtain as follows:

$$\perp_g^{'} = (nr)^{g-1}\perp_1^{'}+\frac{r}{n}\sum_{i=0}^{g-2}(g-1-i)n^{2(g-1-i)}(nr)^i+\frac{(2n-1)r}{2n}\sum_{i=0}^{g-2}n^{2(g-1-i)}(nr)^i+\frac{n^{g-1}r}{2}\sum_{i=0}^{g-2}r^i,$$

where $\perp_1^{'}=nr$.

When $n\neq m-1$, $\perp_g^{'}$ is inductively to obtain as follows:

$$\begin{aligned}\perp_g^{'} &= (mr-r)^{g-1}\perp_1^{'}+\frac{r(m-n)}{m-n-1}\sum_{i=0}^{g-2}(m-1)^{g-i}n^{g-2-i}(mr-r)^i-\frac{r(m-1)(m+n-1)}{2(m-n-1)}\sum_{i=0}^{g-2}n^{2(g-2-i)}\\ &\times (mr-r)^i+\frac{r(m-1)}{2}\sum_{i=0}^{g-2}n^{g-2-i}(mr-r)^i,\end{aligned}$$

where $\perp_1^{'}=mr-r$.

Thus,

$$\begin{aligned}
\perp_g &= \perp'_{g-1} \cdot r \\
&= r \cdot (nr)^{g-1} + \frac{r^2}{n}\sum_{i=0}^{g-3}(g-2-i)n^{2(g-2-i)}(nr)^i + \frac{r^2(2n-1)}{2n}\sum_{i=0}^{g-3}n^{2(g-2-i)}(nr)^i \\
&+ \frac{r^2 \cdot n^{g-2}}{2}\sum_{i=0}^{g-3}r^i, \quad \text{if} \quad n = m-1.
\end{aligned} \tag{48}$$

$$\begin{aligned}
\perp_g &= \perp'_{g-1} \cdot r \\
&= r \cdot (mr-r)^{g-1} + \frac{r^2(m-n)}{m-n-1}\sum_{i=0}^{g-3}(m-1)^{g-1-i}n^{g-3-i}(mr-r)^i - \frac{r^2(m-1)(m+n-1)}{2(m-n-1)}\sum_{i=0}^{g-3}n^{2(g-3-i)} \\
&\times (mr-r)^i + \frac{r^2(m-1)}{2}\sum_{i=0}^{g-3}n^{g-3-i}(mr-r)^i, \quad \text{if} \quad n \neq m-1.
\end{aligned} \tag{49}$$

After calculating $\top_g$ and $\perp_g$, we can obtain

$$\triangle_g = \begin{cases} (g+n-2)n^{g-2}\top_g + n^{g-1}\perp_g, & n = m-1, \\ F\top_g + (m-1)n^{g-2}\perp_g, & n \neq m-1. \end{cases}$$

$$\nabla_g = \begin{cases} 2(g+n-2)n^{g-2}\perp_g + (g+n-2)^2 n^{2(g-2)} \cdot 2n^{g-2}r, & n = m-1, \\ 2F\perp_g + F^2 \cdot 2n^{g-2}r, & n \neq m-1, \end{cases}$$

where $F = \frac{(m-n)(m-1)^{g-1}-(m-1)n^{g-2}}{m-n-1}$.

Using the initial conditions $\sum_{i,j \in D_1^{(1)}} d_{ij} = C_{m-1}^2 \cdot 2r = (m-1)(m-2)r$, $\sum_{i,j\in B_g} d_{ij}$ can be obtained as

$$\sum_{i,j\in B_g} d_{ij} = (m-1)r\sum_{i,j\in B_{g-1}} d_{ij} + \sum_{i,j\in B_g^{(0)}} d_{ij} + (m-1)\triangle_g + \frac{(m-1)(m-2)}{2}\nabla_g.$$

When $n = m-1$, $\sum_{i,j\in B_g} d_{ij}$ is inductively to obtain as follows:

$$\begin{aligned}
\sum_{i,j\in B_g} d_{ij} &= nr\sum_{i,j\in B_{g-1}} d_{ij} + \frac{n^{g-1}r(n^{g-2}+1)[(3n-2)n^{g-2}-1]}{6} \\
&+ (g+n-2)n^{g-1}\top_g + n^g\perp_g + \frac{n(n-1)}{2}[2(g+n-2)n^{g-2}\perp_g + (g+n-2)^2n^{2(g-2)} \cdot 2n^{g-2}r] \\
&= nr\sum_{i,j\in B_{g-1}} d_{ij} + \frac{r(3n-2)}{6n^2}n^{3(g-1)} + \frac{r(n-1)}{2n}n^{2(g-1)} - \frac{r}{6}n^{g-1} + \frac{r(3n-2)}{2n^2}(g-1)n^{3(g-1)} \\
&+ \frac{r(3n-2)(n-1)}{2n^2}n^{3(g-1)} + \frac{r(n-2)}{2n}(g-1)n^{2(g-1)} + \frac{r(n-2)(n-1)}{2n}n^{2(g-1)} \\
&+ \frac{r(n-1)}{n^2}(g-1)^2n^{3(g-1)} + \frac{2r(n-1)^2}{n^2}(g-1)n^{3(g-1)} + \frac{r(n-1)^3}{n^2}n^{3(g-1)} + [(n-1)(g-2)+n^2] \\
&\times [r \cdot (nr)^{g-1} + \frac{r^2}{n}\sum_{i=0}^{g-3}(g-2-i)n^{2(g-2-i)}(nr)^i + \frac{r^2(2n-1)}{2n}\sum_{i=0}^{g-3}n^{2(g-2-i)}(nr)^i + \frac{r^2 \cdot n^{g-2}}{2}\sum_{i=0}^{g-3}r^i].
\end{aligned}$$

$$\sum_{i,j\in B_g} d_{ij} \simeq \begin{cases} g^2n^{3g}, & 0 < r < n^2, \\ g^3 \cdot n^{3g}, & r \geq n^2, \\ g^2 \cdot (nr)^g, & r \geq n^2, \end{cases} \tag{50}$$

where $\sum_{i,j\in B_1} d_{ij} = nr$.

When $n \neq m-1$, $\sum_{i,j\in B_g} d_{ij}$ is inductively to obtain as follows:

$$\begin{aligned}
\sum_{i,j\in B_g} d_{ij} &= (m-1)r \sum_{i,j\in B_{g-1}} d_{ij} + \frac{(m-1)r[(3m-5)n^{3(g-2)} + (3m-6)n^{2(g-2)} - n^{g-2}]}{6} \\
&+ (m-1)F\top_g + [(m-1)^2 n^{g-2} + (m-1)(m-2)F]\bot_g + (m-1)(m-2)F^2 \cdot n^{g-2} r \\
&= (m-1)r \sum_{i,j\in B_{g-1}} d_{ij} + \frac{r(m-1)(3m-5)}{6} n^{3(g-2)} + \frac{r(m-1)(m-2)}{2} n^{2(g-2)} - \frac{r(m-1)}{6} n^{g-2} \\
&+ \frac{r(m-n)(m-1)(3m-5)}{2n^2(m-n-1)}[n^2(m-1)]^{g-1} + \frac{r(m-n)(m-1)(m-3)}{2n(m-n-1)}[n(m-1)]^{g-1} \\
&- \frac{r(m-1)^2(3m-5)}{2(m-n-1)} n^{3(g-2)} - \frac{r(m-1)^2(m-3)}{2(m-n-1)} n^{2(g-2)} \\
&+ \frac{r(m-n)^2(m-1)^3(m-2)}{(m-n-1)^2}[n(m-1)^2]^{g-2} - \frac{2r(m-n)(m-1)^3(m-2)}{(m-n-1)^2}[n^2(m-1)]^{g-2} \\
&+ \frac{r(m-1)^3(m-2)}{(m-n-1)^2} n^{3(g-2)} + [\frac{(m-n)(m-2)}{m-n-1}(m-1)^g - \frac{(n-1)(m-1)^2}{m-n-1} n^{g-2}] \\
&\times [r\cdot(mr-r)^{g-1} + \frac{r^2(m-n)}{m-n-1}\sum_{i=0}^{g-3}(m-1)^{g-1-i} n^{g-3-i}(mr-r)^i - \frac{r^2(m-1)(m+n-1)}{2(m-n-1)} \\
&\times \sum_{i=0}^{g-3} n^{2(g-3-i)}(mr-r)^i + \frac{r^2(m-1)}{2}\sum_{i=0}^{g-3} n^{g-3-i}(mr-r)^i],
\end{aligned}$$

where $\sum_{i,j\in B_g} d_{ij} = \sum_{i,j\in D_g^{(1)}} d_{ij}$.

$$\sum_{i,j\in B_g} d_{ij} \simeq \begin{cases} (m-1)^{2g} n^g, & 0<r<n, \\ g\cdot[(m-1)^2\cdot n]^g, & r=n, \\ [(m-1)^2\cdot r]^g, & r>n, \end{cases} \tag{51}$$

where $\sum_{i,j\in B_1} d_{ij} = (m-1)r$.

References

[1] Meifeng Dai and Danping Zhang. A weighted evolving network with aging-node-deleting and local rearrangements of weights. International Journal of Modern Physics C, 2014, 25 (02): 47-.

[2] Lei. H, Li. T, Ma. YD, Wang. H. Analyzing lattice networks through substructures [J]. Applied Mathematics and Computation, 2018, 329: 297-314.

[3] Lucas S. Flores; Marco A. Amaral; Mendeli H. Vainstein; Heitor C. M. Fernandes. Cooperation in regular lattices [J]. Chaos, Solitons & Fractals, 2022£º112744.

[4] Robert M, Ziff. Explosive Growth in Biased Dynamic Percolation on Two-Dimensional Regular Lattice Networks [J]. Physical Review Letters 2009, 103 (04): 045701 (1-4).

[5] Garza-López, Roberto A; Linares, Anthony; Yoo, Alice; Evans, Greg; Kozak, John J. Invariance relations for random walks on simple cubic lattices [J]. Chemical Physics Letters, 2006, 421 (NO. 1-3): 287-294.

[6] Garza-López, Roberto A; Kozak, John J. Invariance relations for random walks on square-planar lattices [J]. Chemical Physics Letters, 2005, 406 (NO. 1-3): 38-43.

[7] Garza-López, Roberto A; Kozak, John J. Invariance relations for random walks on hexagonal lattices [J]. Chemical Physics Letters, 2003, 371 (03): 365-370.

[8] John J. Kozak and V. Balakrishnan. Analytic expression for the mean time to absorption for a random walker on the Sierpinski gasket [J]. Physical Review. E, 2002, 65 (02): 021105.

[9] V. Balakrishnan and John J. Kozak. Analytic expression for the mean time to absorption for a random walker on the Sierpinski fractal. III. The effect of non-nearest-neighbor jumps [J]. PHYSICAL REVIEW E, 2013, 88 (05): 52139-52139.

[10] Jonathan L Bentz; John W Turner; John J Kozak. Analytic expression for the mean time to absorption for a random walker on the Sierpinski gasket. II. The eigenvalue spectrum [J]. Physical review. E, Statistical, nonlinear, and soft matter physics, 2010, 82 (01-01): 011137.

[11] John J. Kozak and V. Balakrishnan. Exact Formula for the Mean Length of a Random Walk on the Sierpinski Tower [J]. International Journal of Bifurcation and Chaos, 2002, 12 (11): 2379-2385.

[12] Zhang, ZZ; Zhou, SG; Xie, WL; Chen, LC; Lin, Y; Guan, JH. Standard random walks and trapping on the Koch network with scale-free behavior and small-world effect [J]. Physical review. E, Statistical, nonlinear, and soft matter physics, 2009, 79 (06-01): 061113.

[13] Zhang, JY; Sun, WG; Chen, GR. Exact scaling for the mean first-passage time of random walks on a generalized Koch network with a trap [J]. Chinese Physics B, 2012, 21 (03): 038901.

[14] Su, Jing; Zhang, Mingjun; Yao, Bing. The Structure and First-Passage Properties of Generalized Weighted Koch Networks [J]. Entropy, 2022, 24 (03): 409.

[15] Zhang, JY and Sun, WG. The structural properties of the generalized Koch network [J]. Journal of Statistical Mechanics: Theory and Experiment, 2010, 2010 (07): P07011.

[16] E. Agliari. Exact mean first-passage time on the T-graph [J]. Physical review, E. Statistical, nonlinear, and soft matter physics, 2008, 77 (01-01): 011128 (1-6).

[17] Lin, Yuan; Zhang, Zhongzhi; Wu, Bin. Determining mean first-passage time on a class of treelike regular fractals [J]. Physical Review E, 2010, 82 (03): 031140.

[18] Zhang, Zhongzhi; Wu, Bin; Chen, Guanrong. Complete spectrum of the stochastic master equation for random walks on treelike fractals [J]. Europhysics Letters, 2011, 96 (4): 2510-2513.

[19] Julaiti, Alafate; Wu, Bin; Zhang, Zhongzhi. Eigenvalues of normalized Laplacian matrices of fractal trees and dendrimers: Analytical results and applications [J]. Journal of Chemical Physics, 2013, 138 (20): 204116.

[20] Peng J and Xu G. Efficiency analysis of diffusion on T-fractals in the sense of random walks [J]. The Journal of Chemical Physics, 2014, 140 (13): 134102.

[21] Roberts AP and Haynes CP. Global first-passage times of fractal lattices [J]. Physical review, E, 2008, 78 (04-01): 041111 (1-9).

[22] S. N Dorogovtsev, A. V Goltsev, J. F. F Mendes. Pseudofractal scale-free web [J]. PHYSICAL REVIEW E, 2002, 65 (6): 066122.

[23] Zhang, ZZ; Qi, Y; Zhou, SG; Xie, WL; Guan, JH. Exact solution for mean first-passage time on a pseudofractal scale-free web [J]. Physical review, E, 2009, 79 (02-01): 021127 (1-6).

[24] Peng, JH; Agliari, E; Zhang, ZZ. Exact calculations of first-passage properties on the pseudofractal scale-free web [J]. CHAOS, 2015, 25 (07): 073118.

[25] Wu, B. The average trapping time on the weighted pseudofractal scale-free web [J]. Journal of Statistical Mechanics: Theory and Experiment, 2020, 2020 (04): 043209.

[26] Zhang, ZZ; Chen, LC; Zhou, SG; Fang, L; Guan, JH; Zou, T. Analytical solution of average path length for Apollonian networks [J]. Physical review, E, 2008, 77 (01-02): 017102 (1-4).

[27] José S Andrade; Hans J Herrmann; Roberto F S Andrade; Luciano R da Silva. Apollonian networks: simultaneously scale-free, small world, euclidean, space filling, and with matching graphs [J]. Physical review letters, 2005, 94 (01): 018702.

[28] Zhang, Zhongzhi; Guan, Jihong; Xie, Wenlei; Qi, Yi; Zhou, Shuigeng. Random walks on the Apollonian network with a single trap [J]. EPL, 2009, 86 (01): 10006.

[29] Zhang, ZZ; Rong, LL; Zhou, SG. Evolving Apollonian networks with small-world scale-free topologies [J]. Physical review, E, 2006, 74 (04-02): 6105 (1-9).

[30] Zhang, ZZ; Lin, YA; Gao, SY; Zhou, SG; Guan, JH; Li, M. Trapping in scale-free networks with hierarchical organization of modularity [J]. Physical review, E, 2009, 80 (05-01): 051120 (1-10).

[31] Yang, Y; Lin, Y; Zhang, Z. Random walks in modular scale-free networks with multiple traps [J]. Physical review, E, 2012, 85 (01-01): 011106 (1-10).

[32] Niu, Min; Shao, Mengjun. The asymptotic formula on average weighted path length for scale-free modular network [J]. Modern Physics Letters B, 2021, 35 (18): 2150298.

[33] Bentz J L, Hosseini F N, Kozak J J. Influence of geometry on light harvesting in dendrimeric systems [J]. Chemical Physics Letters, 2003, 370 (3-4); 319-326

[34] Li L, Guan J, Zhou S. Mean first passage time for random walk on dual structure of dendrimer [J]. Physica A Statistical Mechanics & Its Applications, 2014, 415: 463-472.

[35] Argyrakis P, Kopelman R. Random walks and reactions on dendrimer structures [J]. Chemical Physics, 2000, 261 (3): 391-398.

[36] Ma, Fei; Wang, Xiaomin; Wang, Ping; Luo, Xudong. Random walks on the generalized Vicsek fractal [J]. EPL, 2021, 133 (04): 40004.

[37] Dolgushev, Maxim; Liu, Hongxiao; Zhang, Zhongzhi. Extended Vicsek fractals: laplacian spectra and their applications [J]. Physical Review E, 2016, 94 (05): 052501.

[38] A. Blumen; Ch. von Ferber; A. Jurjiu; Th. Koslowski. Generalized Vicsek Fractals: Regular Hyperbranched Polymers [J]. Macromolecules, 2004, 37 (02): 638-650.

[39] Zhongzhi Zhang; Shuigeng Zhou; Lichao Chen; Ming Yin; Jihong Guan. Exact solution of mean geodesic distance for Vicsek fractals [J]. Journal of physics, A. Mathematical and theoretical, 2008, 41 (48): 485102.

[40] Zhang, ZZ; Wu, B; Zhang, HJ; Zhou, SG; Guan, JH; Wang, ZG. Determining global mean-first-passage time of random walks on Vicsek fractals using eigenvalues of Laplacian matrices [J]. Physical Review E, 2010, 81 (03-01): 031118.

[41] D. Y Yan, C. Gao, and H. Frey. Hyperbranched Polymers [J]. Chemical Society Reviews, 2011: 251-271.

[42] Xu, Ming; Xu, Chuanyun; Wang, Huan; Deng, Congzheng; Cao, KeiFei. Analytical controllability of deterministic scale-free networks and Cayley trees [J]. European Physical Journal B, 2015, 88 (07): 168.

[43] Lin Y, Zhang Z. Influence of trap location on the efficiency of trapping in dendrimers and regular hyperbranched polymers [J]. Journal of Chemical Physics, 2013, 138 (9): 088701-R.

[44] Wu B, Lin Y, Zhang ZZ, Chen, GR. Trapping in dendrimers and regular hyperbranched polymers [J]. Journal of Chemical Physics, 2012, Vol. 137 (04): 044903.

[45] Marc Barthélemy; Alain Barrat; Romualdo Pastor-Satorras; Alessandro Vespignani. Characterization and modeling of weighted networks [J]. Physica A: Statistical Mechanics and its Applications, 2005, 346 (1-2): 34-43.

[46] Carletti T and Righi S. Weighted Fractal Networks [J]. Physica A: Statal Mechanics and its Applications, 2009, 389 (10): 2134-2142.

[47] Dai, MF; Zong, Y; He, JJ; Sun, Y; Shen, CY; Su, WY. The trapping problem of the weighted scale-free treelike networks for two kinds of biased walks [J]. Chaos, 2018, 28 (11): 113115.

[48] Dai, MF; Liu, J; Li, XY. Trapping time of weighted-dependent walks depending on the weight factor [J]. Chaos Solitons & Fractals, 2014, 60: 49-55.

[49] Ann E Krause; Kenneth A Frank; Doran M Mason; Robert E Ulanowicz; William W Taylor. Compartments revealed in food-web structure [J]. Nature, 2003, 426 (6964): 282.

[50] Chen J, Dai MF, Wen ZX, Xi LF. Trapping on modular scale-free and small-world networks with multiple hubs [J]. Physica A Statistical Mechanics & Its Applications, 2014, 393: 542-552.

[51] Shlomo Havlin; Daniel Ben-Avraham. Diffusion in disordered media [J]. Advances in physics, 2002, 51 (01): 187-292.

[52] Lloyd, Alun L. How viruses spread among computers and people [J]. Science, 2001, 292 (5520): 1316-1317.

[53] Shlesinger MF. Mathematical physics: search research [J]. Nature, 2006, 443 (7109): 281.

[54] Dandan Ye, Meifeng Dai, Yanqiu Sun, Shuxiang Shao, Qi Xie. Average receiving scaling of the weighted polygon Koch networks with the weight-dependent walk [J]. Physica A - Statistical mechanics and its applications, 2016, 458: 1-8.

[55] Stepanenko IA, Kompanets VO, Chekalin SV, et al. Photosynthetic light-harvesting complexes: fluorescent and absorption spectroscopy under two-photon (1200-1500 nm) and one-photon (600-750 nm) excitation by laser femtosecond pulses [J]. proc spie, 2011, 7994 (2): 170-187.

[56] Bentz JL, Kozak JJ. Influence of geometry on light harvesting in dendrimeric systems. II. nth-nearest neighbor effects and the onset of percolation [J]. Journal of Luminescence, 2006, 121 (1): 62-74.

[57] Agliari E. Trapping of Continuous-Time Quantum walks on Erdos-Renyi graphs [J]. Physica A Statistical Mechanics & Its Applications, 2011, 390 (11): 1853-1860.

[58] Sokolov IM, Mai J, Blumen A. Paradoxal Diffusion in Chemical Space for Nearest-Neighbor Walks over Polymer Chains [J]. Physical Review Letters, 1997, 79 (5): 857-860.

[59] Blumen, A. Energy transfer as a random walk on regular lattices [J]. Journal of Chemical Physics, 1981, 75 (2): 892-907.

[60] Dai, Meifeng; Dai, Changxi; Wu, Huiling; Wu, Xianbin; Feng, Wenjing; Su, Weiyi. The trapping problem and the average shortest weighted path of the weighted pseudofractal scale-free networks [J]. INTERNATIONAL JOURNAL OF MODERN PHYSICS C, 2019, 30 (01): 1950010.

[61] Dai, Meifeng; Ju, Tingting; Zong, Yue; Fie, Jiaojiao; Shen, Chunyu; Su, Weiyi. TRAPPING PROBLEM OF THE WEIGHTED SCALE-FREE TRIANGULATION NETWORKS FOR BIASED WALKS [J]. FRACTALS-COMPLEX GEOMETRY PATTERNS AND SCALING IN NATURE AND SOCIETY, 2019, 27 (03).

[62] Dai, Meifeng; Xie, Qi; Xi, Lifeng. TRAPPING ON WEIGHTED TETRAHEDRON KOCH NETWORKS WITH SMALL-WORLD PROPERTY [J]. FRACTALS-COMPLEX GEOMETRY PATTERNS AND SCALING IN NATURE AND SOCIETY, 2014, 22 (1-2): 1-9.

[63] Zhang ZZ, Li H, Yi Y. Anomalous behavior of trapping in extended dendrimers with a perfect trap [J]. The Journal of Chemical Physics, 2015, 143 (6): 064901.

[64] Niu M, Song SS. Scaling of average weighted shortest path and average receiving time on the weighted Cayley networks [J]. Physica A: Statal Mechanics and its Applications, 2018, 506 (01): 707-717.

[65] S. Boccaletti, et al. Complex networks: Structure and dynamics [J]. Physics Reports, 2006, 424 (4-5): 175-308.

[66] Sun Y, Dai MF, Xi LF. Scaling of average weighted shortest path and average receiving time on weighted hierarchical networks [J]. Physica, A. Statistical mechanics and its applications, 2014, 407: 110-118.

[67] Dai MF, Chen DD, Dong YJ, Liu JE. Scaling of average receiving time and average weighted shortest path on weighted Koch networks [J]. Physica A - Statistical mechanics and its applications, 2012, 391 (23): 6165-6173.

[68] Dandan Ye, Song Liu, Jia Li, Fei Zhang, Changling Han, Wei Chen, Yingze Zhang. Scaling of average receiving time on weighted polymer networks with some topological properties [J]. Scientific Reports. 2017, 07 (01): 2128.

[69] Fei Zhang, Dandan Ye, Changling Han, Wei Chen, Yingze Zhang. The average weighted receiving time with weight-dependent walk on a family of double-weighted polymer networks [J]. International Journal of Modern Physics C: Computational Physics & Physical Computation. 2019, 30 (08): 1950063.

[70] Meifeng Dai, Dandan Ye, Jie Hou, Lifeng Xi, Weiyi Su. Average weighted trapping time of the node- and edge-weighted fractal networks [J]. Communications in nonlinear science and numerical simulation, 2016, 39 (10): 209-219.

[71] Dandan Ye, Meifeng Dai, Yu Sun, Weiyi Su. Average weighted receiving time on the non-homogeneous double-weighted fractal networks [J]. Physica A - Statistical mechanics and its applications, 2017, 473: 390-402.

[72] Condamin S, Bénichou O, Tejedor V, Voituriez R, Klafter J. First-passage times in complex scale-invariant media [J]. Nature 2007; 450 (7166): 77-80.

[73] Meifeng Dai, Dandan Ye, Jie Hou, Lifeng Xi. Scaling of average weighted receiving time on double-weighted koch networks [J]. Fractals-Complex Geometry Patterns and Scaling in Nature and Society, 2015, 23 (02): 1550011.